“Some New Variance Reduction Ideas in Simulation”

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1 Stratified sampling in Simulation

\[ \theta = E[X] \]
usual: \( \bar{X}, \ Var = \text{Var}(X)/n \)

\[ E[X] = \sum_{i=1}^{k} E[X|Y = y_i]p_i \]

\( np_i \) runs conditional on \( Y = y_i \)

\[ \mathcal{E} = \sum_{i=1}^{k} \bar{X}_i p_i \]

\[ \text{Var}(\mathcal{E}) = \sum_{i=1}^{k} p_i^2 \text{Var}(\bar{X}_i) \]

\[ = \frac{1}{n} \sum_{i=1}^{k} p_i \text{Var}(X|Y = y_i) \]

\[ = \frac{1}{n} E[\text{Var}(X|Y)] \]
$D$: sum of the delays of all arrivals by $t$

$$E[D] = \sum_{j=0}^{m} E[D|N(t) = j]p_j + E[D|N(t) > m] \tilde{P}_m$$

1. $N(t) = 1$. Generate $U$; let $\mathcal{A} = \{tU\}$.
2. Generate $S$; calculate $D_1$.
3. Let $N(t) = N(t) + 1$.
4. Generate $U$, add $tU$ to $\mathcal{A}$
5. Generate $S$; calculate $D_{N(t)}$
6. If $N < m$ return to Step 3.
7. Generate $N(t)|N(t) > m$:
   $A_i, S_i, i = m + 1, \ldots, N(t) : D_{>m}$

$$\mathcal{E} = \sum_{j=0}^{m} D_j e^{-\lambda t} (\lambda t)^j / j! + D_{>m} (1 - \sum_{j=0}^{m} e^{-\lambda t} (\lambda t)^j / j!)$$
Theorem \( \Var(\mathcal{E}) \leq \Var(D) \)

Proof: Generate \( D \) by

1. Generating \( N' =_{d} N(t) \mid N(t) > m \)
2. Generate \( A_1, \ldots, A_{N'}, \text{iid } U(0, t) \)
3. Generate \( S_1, \ldots, S_{N'} \)
4. Generate \( N(t) \)
5. If \( N(t) = j \leq m \), use \( A_1, \ldots, A_j \) and \( S_1, \ldots, S_j : D = D_j \).
6. If \( N(t) > m \), use \( A_1, \ldots, A_{N'} \) and \( S_1, \ldots, S_{N'} : D = D_{>m} \).

Conditioning on \( N(t) \), gives

\[
E[D \mid N', A_1, \ldots, A_{N'}, S_1, \ldots, S_{N'}] = \mathcal{E}
\]
\[ H = h(X_1, \ldots, X_n); \ h(0, \ldots, 0) = 0 \]
\[ \alpha = E[H] \]

Given \( \Theta = \theta \)

**Model 1:** \( X_i \) independent Poisson:
\[ E[X_i] = \theta \lambda_i \]

**Model 2:** \( X_i \) independent Bernoulli:
\[ E[X_i] = \theta p_i \]
\[ m = ? \]
\[ E[X|\Theta] = \text{Var}(X|\Theta) = \lambda \Theta \]
\[ E[X] = \lambda E[\Theta] \]
\[ \text{Var}(X) = \lambda E[\Theta] + \lambda^2 \text{Var}(\Theta) \]
\[ m = E[X] + k\sqrt{\text{Var}(X)} \]

How to generate \( X|X > m \)?

inverse transform
**Theorem:** \( \text{Var}(\hat{\alpha}) \leq \text{Var}(H) \)

**Pf:** Generate \( H \) as follows:

1. Generate \( I = I_1, \ldots \)
2. Generate \( Y \) distributed as \( X | X > m \)
3. Generate \( X \)
4. If \( X \leq m \), let \( X_i \) be number of \( I_1, \ldots, I_X \) equal to \( i \)
5. If \( X > m \), let \( X_i \) be number of \( I_1, \ldots, I_Y \) equal to \( i \)

6. \( H = h(X_1, \ldots, X_n) \)

\[
E[H|I] = \sum_{i=1}^{m} E[H|I, X = i]P_i + E[H|I, X > m]P_{>m} = \hat{\alpha}
\]
Bernoulli model

Case 1: $X_1, \ldots, X_n$ exchangeable

$$X = \sum_{i=1}^{n} X_i$$

$$P_j = P(X = j)$$

cond iid:

$$P_j = \int \binom{n}{j} (\theta p)^j (1 - \theta p)^{n-j} dF(\theta)$$

$$\alpha = E[H] = \sum_{j=0}^{n} E[H|X = j] P_j$$

1. Generate random permutation $I_1, \ldots, I_n$
2. $j = 0$, all $x_i = 0$
3. $H_j = h(x_1, \ldots, x_n)$
4. $j = j + 1$, $x_{I_j} = 1$
5. If $j < n$ return to 3
6. $H_n = h(1, \ldots, 1)$

$$\hat{\alpha} = \sum_{j=0}^{n} H_j P_j$$
Special Case: \( h \) binary and increasing

\[
\hat{\alpha} = \sum_{j = K}^{n} P_j
\]

\[N = \min(k : (I_1, \ldots, I_k) \text{ is min path})\]

\[
\hat{\alpha} = \sum_{j \geq N} P_j
\]
Example 1: bridge system with $p(\theta) = \theta$

$P\{N = 2\} = 1/5, P\{N = 3\} = 3/5, P\{N = 4\} = 1/5$

$$\text{Est} = \begin{cases} 
\sum_{j=2}^{5} b_j, & \text{with prob. } 1/5 \\
\sum_{j=3}^{5} b_j, & \text{with prob. } 3/5 \\
\sum_{j=4}^{5} b_j, & \text{with prob. } 1/5 
\end{cases}$$

<table>
<thead>
<tr>
<th>distribution</th>
<th>$\pi$</th>
<th>$\pi(1-\pi)$</th>
<th>Var(Est)</th>
</tr>
</thead>
<tbody>
<tr>
<td>uniform $(0, 1)$</td>
<td>.5</td>
<td>.25</td>
<td>.011</td>
</tr>
<tr>
<td>deterministic</td>
<td>.5</td>
<td>.25</td>
<td>.039</td>
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<tr>
<td>deterministic</td>
<td>.95</td>
<td>.9948</td>
<td>.00008</td>
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</table>
\[ P(X_i = 1|\Theta = \theta) = \theta p_i \]

- choose \( p : p_i \leq p \)
- \( X_i = 1 \) if \( i \) passes 2 stages
- \( i \) passes stage 1 w. prob. \( p_i/p \)
- \( i \) passes stage 2 w. prob \( p\Theta \)
- Generate stage 1
- Use exchangeable for conditional system

Computation: need

\[ P_j(r) = \int \binom{r}{j} (\theta p)^j (1 - \theta p)^{r-j} dF(\theta) \]
Special Case: \( P(\Theta = 1) = 1 \)

- choose \( p \)

- \( p_i < p \)
  
  \[ X_i = 1 \] if \( i \) passes 2 stages
  
  \( i \) passes stage 1 w. prob. \( p_i/p \)
  
  \( i \) passes stage 2 w. prob \( p \)

- \( p_i > p : \)
  
  \[ X_i = 1 \] if \( i \) passes either stage
  
  \( i \) fails stage 1 w. prob. \( (1-p_i)/(1-p) \)
  
  \( i \) passes stage 2 w. prob. \( p \)

- Generate stage 1

- Use exchangeable for conditional system
Example m-of-100: \( p_i = .4 + .002i, \ i = 1, \ldots, 100. \)

\( p = .5 \)

<table>
<thead>
<tr>
<th>( m )</th>
<th>( \pi )</th>
<th>( \pi(1 - \pi) )</th>
<th>( \text{Var}(\hat{\pi}) )</th>
</tr>
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<tbody>
<tr>
<td>30</td>
<td>0.999987</td>
<td>1.3 \times 10^{-5}</td>
<td>5.15 \times 10^{-10}</td>
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<td>7.72 \times 10^{-4}</td>
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<td>60</td>
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<td>.00045</td>
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Additional Variance Reduction: Antithetic variables in Stage 1
\[ \theta = E[g(U_1, \ldots, U_n)] \\
= \int_0^1 \int_0^1 \cdots \int_0^1 g(x_1, x_2, \ldots, x_n) \, dx_1 \, dx_2 \cdots \, dx_n \]

Now,
\[ E[\text{Var}(g(U_1, \ldots, U_n) | \prod_{i=1}^n U_i)] \] often small.

(a) generate \( U_1, \ldots, U_n \) conditional on \( \prod_{i=1}^n U_i \)
(b) generate the value of \( \prod_{i=1}^n U_i \) in a stratified fashion.

\[ T_j = \sum_{i=1}^j - \log(U_i) = - \log(U_1 \cdots U_j) \]

Generate
\[ T_n = - \log(U_1 \cdots U_n) \]

Generate \( V_1, \ldots, V_{n-1} \), order them
\[ V_{(1)} < V_{(2)} < \ldots < V_{(n-1)} \]
\[ T_n V_{(j)} = -\log(U_1 \cdots U_j) \]
\[ = T_n V_{(j-1)} - \log(U_j) \]

Therefore,
\[ U_j = e^{-T_n[V_{(j)} - V_{(j-1)}]}, \quad j = 1, \ldots, n \]

To generate \( T_n \) on run \( k \):
\[ G_n^{-1}\left( \frac{U + k - 1}{n} \right). \]
\[ \theta = E[g(B)] \]

where \( B \) is equally likely to be any of the \( \binom{n}{k} \) subsets of \( S = \{1, 2, \ldots, n\} \)

**usual:** independent choices of \( B \)

**idea:** choose next subset from unchosen elements

**works:** when \( g \) is monotone