# “Some New Variance Reduction Ideas in Simulation”

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## 1 Stratified sampling in Simulation

$\theta = E[X]$

usual: $\bar{X}$, $\quad \text{Var} = \text{Var}(X)/n$

$$E[X] = \sum_{i=1}^{k} E[X|Y = y_i]p_i$$

$np_i$ runs conditional on $Y = y_i$

$$\mathcal{E} = \sum_{i=1}^{k} \bar{X}_i p_i$$

$$\begin{aligned} \text{Var}(\mathcal{E}) &= \sum_{i=1}^{k} p_i^2 \text{Var}(\bar{X}_i) \\ &= \frac{1}{n} \sum_{i=1}^{k} p_i \text{Var}(X|Y = y_i) \\ &= \frac{1}{n} E[\text{Var}(X|Y)] \end{aligned}$$

$D$: sum of the delays of all arrivals by $t$

$$E[D] = \sum_{j=0}^{m} E[D|N(t) = j]p_j + E[D|N(t) > m]\bar{P}_m$$

1. $N(t) = 1$. Generate $U$; let $\mathcal{A} = \{tU\}$.
2. Generate $S$; calculate $D_1$.
3. Let $N(t) = N(t) + 1$.
4. Generate $U$, add $tU$ to $\mathcal{A}$
5. Generate $S$; calculate $D_{N(t)}$
6. If $N < m$ return to Step 3.
7. Generate $N(t)|N(t) > m$ :
   $A_i, S_i, i = m+1, \ldots, N(t)$ : $D_{>m}$

$$\mathcal{E}$$

$$= \sum_{j=0}^{m} D_j e^{-\lambda t}(\lambda t)^j/j! + D_{>m}(1 - \sum_{j=0}^{m} e^{-\lambda t}(\lambda t)^j/j!)$$

**Theorem** $\text{Var}(\mathcal{E}) \leq \text{Var}(D)$

**Proof:** Generate $D$ by

**1.** Generating $N' =_d N(t)|N(t) > m$
**2.** Generate $A_1, \ldots, A_{N'}$, iid $U(0,t)$
**3.** Generate $S_1, \ldots, S_{N'}$
**4.** Generate $N(t)$
**5.** If $N(t) = j \leq m$, use $A_1, \ldots, A_j$ and $S_1, \ldots, S_j$ : $D = D_j$.
**6.** If $N(t) > m$, use $A_1, \ldots, A_{N'}$ and $S_1, \ldots, S_{N'}$ : $D = D_{>m}$.

Conditioning on $N(t)$, gives

$$E[D|N', A_1, \ldots, A_{N'}, S_1, \ldots, S_{N'}] = \mathcal{E}$$

$$H = h(X_1, \ldots, X_n);\ h(0, \ldots, 0) = 0$$

$$\alpha = E[H]$$

Given $\Theta = \theta$

**Model 1:** $X_i$ independent Poisson:
$E[X_i] = \theta\lambda_i$

**Model 2:** $X_i$ independent Bernoulli:
$E[X_i] = \theta p_i$

$$m = ?$$

$$E[X|\Theta] = \mathrm{Var}(X|\Theta) = \lambda\Theta$$

$$E[X] = \lambda E[\Theta]$$

$$\mathrm{Var}(X) = \lambda E[\Theta] + \lambda^2 \mathrm{Var}(\Theta)$$

$$m = E[X] + k\sqrt{\mathrm{Var}(X)}$$

How to generate $X|X > m$?

inverse transform

**Theorem:** $\mathrm{Var}(\hat{\alpha}) \le \mathrm{Var}(H)$

Pf: Generate $H$ as follows:

1. Generate $\mathbf{I} = I_1, \ldots$
2. Generate $Y$ distributed as $X | X > m$
3. Generate $X$
4. If $X \le m$, let $X_i$ be number of $I_1, \ldots, I_X$ equal to $i$
5. If $X > m$, let $X_i$ be number of $I_1, \ldots, I_Y$ equal to $i$
6. $H = h(X_1, \ldots, X_n)$

$E[H|\mathbf{I}]$

$$= \sum_{i=1}^{m} E[H|\mathbf{I}, X = i] P_i + E[H|\mathbf{I}, X > m] P_{>m}$$
$$= \hat{\alpha}$$

## Bernoulli model

Case 1: $X_1, \ldots, X_n$ exchangeable

$$X = \sum_{i=1}^{n} X_i$$

$$P_j = P(X = j)$$

cond iid:

$$P_j = \int \binom{n}{j} (\theta p)^j (1 - \theta p)^{n-j} dF(\theta)$$

$$\alpha = E[H] = \sum_{j=0}^{n} E[H | X = j] P_j$$

1. Generate random permutation $I_1, \ldots, I_n$
2. $j = 0$, all $x_i = 0$
3. $H_j = h(x_1, \ldots, x_n)$
4. $j = j + 1,\ x_{I_j} = 1$
5. If $j < n$ return to 3
6. $H_n = h(1, \ldots, 1)$

$$\hat{\alpha} = \sum_{j=0}^{n} H_j P_j$$

**Special Case:** $h$ binary and increasing

$$\hat{\alpha} = \sum_{j=K}^{n} P_j$$

$$N = \min(k : (I_1, \ldots, I_k) \text{ is min path})$$

$$\hat{\alpha} = \sum_{j \geq N} P_j$$

**Example 1:** bridge system with $p(\theta) = \theta$

$P\{N = 2\} = 1/5$, $P\{N = 3\} = 3/5$, $P\{N = 4\} = 1/5$

$$\text{Est} = \begin{cases} \Sigma_{j=2}^{5} b_j, & \text{with prob. } 1/5 \\ \Sigma_{j=3}^{5} b_j, & \text{with prob. } 3/5 \\ \Sigma_{j=4}^{5} b_j, & \text{with prob. } 1/5 \end{cases}$$

| **distribution** | $\pi$ | $\pi(1-\pi)$ | Var(Est) |
|---|---|---|---|
| uniform$(0, 1)$ | .5 | .25 | .011 |
| deterministic .5 | .5 | .25 | .039 |
| deterministic .7 | .8016 | .1590 | .0213 |
| deterministic .9 | .9785 | .0211 | .0009 |
| deterministic .95 | .9948 | .0052 | .00008 |

$$P(X_i = 1|\Theta = \theta) = \theta p_i$$

- choose $p$ : $\quad p_i \leq p$
- $X_i = 1$ if $i$ passes 2 stages
- $i$ passes stage 1 w. prob. $p_i/p$
- $i$ passes stage 2 w. prob $p\Theta$
- Generate stage 1
- Use exchangeable for conditional system

Computation: need

$$P_j(r) = \int \binom{r}{j} (\theta p)^j (1 - \theta p)^{r-j} dF(\theta)$$

**Special Case:** $P(\Theta = 1) = 1$

- choose $p$
- $p_i < p$

  $X_i = 1$ if $i$ passes 2 stages

  $i$ passes stage 1 w. prob. $p_i/p$

  $i$ passes stage 2 w. prob $p$
- $p_i > p$ :

  $X_i = 1$ if $i$ passes either stage

  $i$ fails stage 1 w. prob. $(1-p_i)/(1-p)$

  $i$ passes stage 2 w. prob. $p$
- Generate stage 1
- Use exchangeable for conditional system

**Example m-of-100:** $p_i = .4 + .002i$, $i = 1, \ldots, 100$. $p = .5$

| $m$ | $\pi$ | $\pi(1-\pi)$ | $\mathrm{Var}(\hat{\pi})$ |
|---|---|---|---|
| 30 | 0.999987 | $1.3 \times 10^{-5}$ | $5.15 \times 10^{-10}$ |
| 35 | 0.999227 | $7.72 \times 10^{-4}$ | $9.20 \times 10^{-7}$ |
| 40 | 0.98396 | .0158 | .00017 |
| 45 | 0.8701 | .1130 | .00412 |
| 50 | 0.5476 | .2477 | .0140 |
| 55 | 0.1889 | .1532 | .0067 |
| 60 | 0.0291 | .0283 | .00045 |

Additional Variance Reduction: Antithetic variables in Stage 1

## 3 Multi Dimensional Integrals of Monotone Functiona

$$\begin{aligned}\theta &= E[g(U_1, \ldots, U_n)] \\ &= \int_0^1 \int_0^1 \cdots \int_0^1 g(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \cdots dx_n\end{aligned}$$

Now,
$E[\mathrm{Var}(g(U_1, \ldots, U_n) \mid \Pi_{i=1}^n U_i)]$ often small.

(a) generate $U_1, \ldots, U_n$ conditional on $\Pi_{i=1}^n U_i$
(b) generate the value of $\Pi_{i=1}^n U_i$ in a stratified fashion.

$$T_j = \sum_{i=1}^{j} -\log(U_i) = -\log(U_1 \cdots U_j)$$

Generate

$$T_n = -\log(U_1 \cdots U_n)$$

Generate $V_1, \ldots, V_{n-1}$, order them

$$V_{(1)} < V_{(2)} < \ldots < V_{(n-1)}$$

$$\begin{aligned} T_n V_{(j)} &= -\log(U_1 \cdots U_j) \\ &= T_n V_{(j-1)} - \log(U_j) \end{aligned}$$

Therefore,

$$U_j = e^{-T_n[V_{(j)} - V_{(j-1)}]}, \quad j = 1, \ldots, n$$

To generate $T_n$ on run $k$:

$$G_n^{-1}\left(\frac{U + k - 1}{n}\right).$$

$$\theta = E[g(B)]$$

where $B$ is equally likely to be any of the $\binom{n}{k}$ subsets of $S = \{1, 2, \ldots, n\}$

**usual:** independent choices of $B$

**idea:** choose next subset from unchosen elements

**works:** when $g$ is monotone