Quantum Network Coding
-- the butterfly & beyond

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Plan:

- Motivating example

    The classical 2-pair communication problem

        the solution in the butterfly network

    Prior work in quantum setting

    Our scenario & results -- unlike the classical case rerouting is optimal

- Results for more general networks

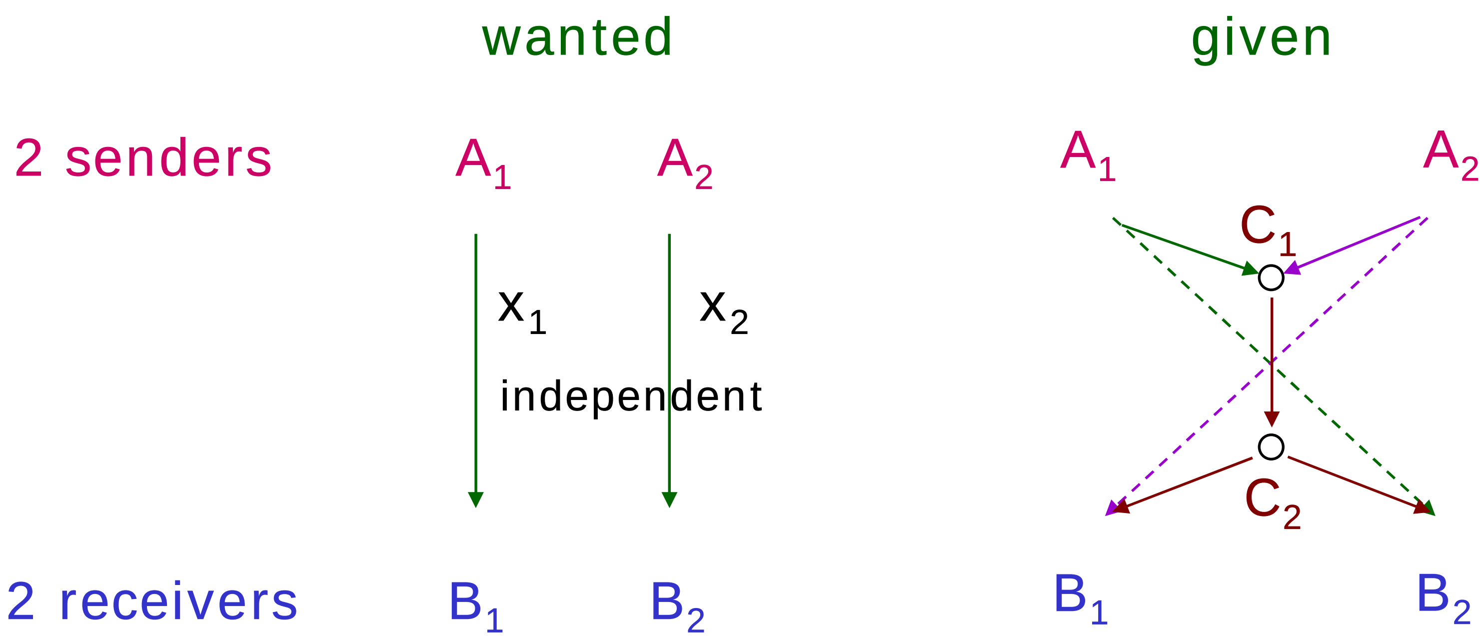
    Optimality of rerouting in

    (a) k-pair comm in shallow networks

    (b) 2-pair comm in any "back-assisted" network

Motivating example : the butterfly network

The 2-pair comm problem (classical):



Assumption: "network" (all 7 channels) called as a package

Qn: how to "best" communicate from A_1 to B_1 and A_2 to B_2?

Motivating example : the butterfly network

For independent bits x_1, x_2, a "best" defines itself

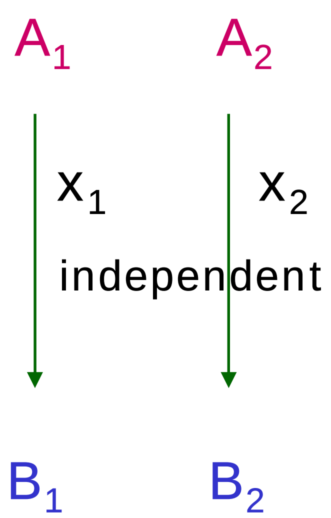
B_1, B_2 both get both x_1, x_2

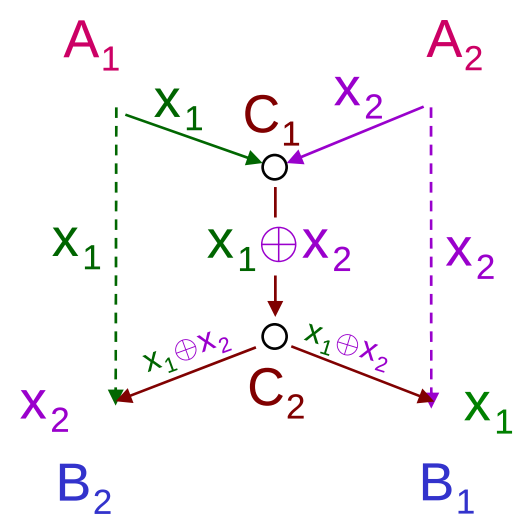
wanted

given

2 senders

2 receivers





Assumption: "network" (all 7 channels) called as a package

Qn: how to "best" communicate from A_1 to B_1 and A_2 to B_2?

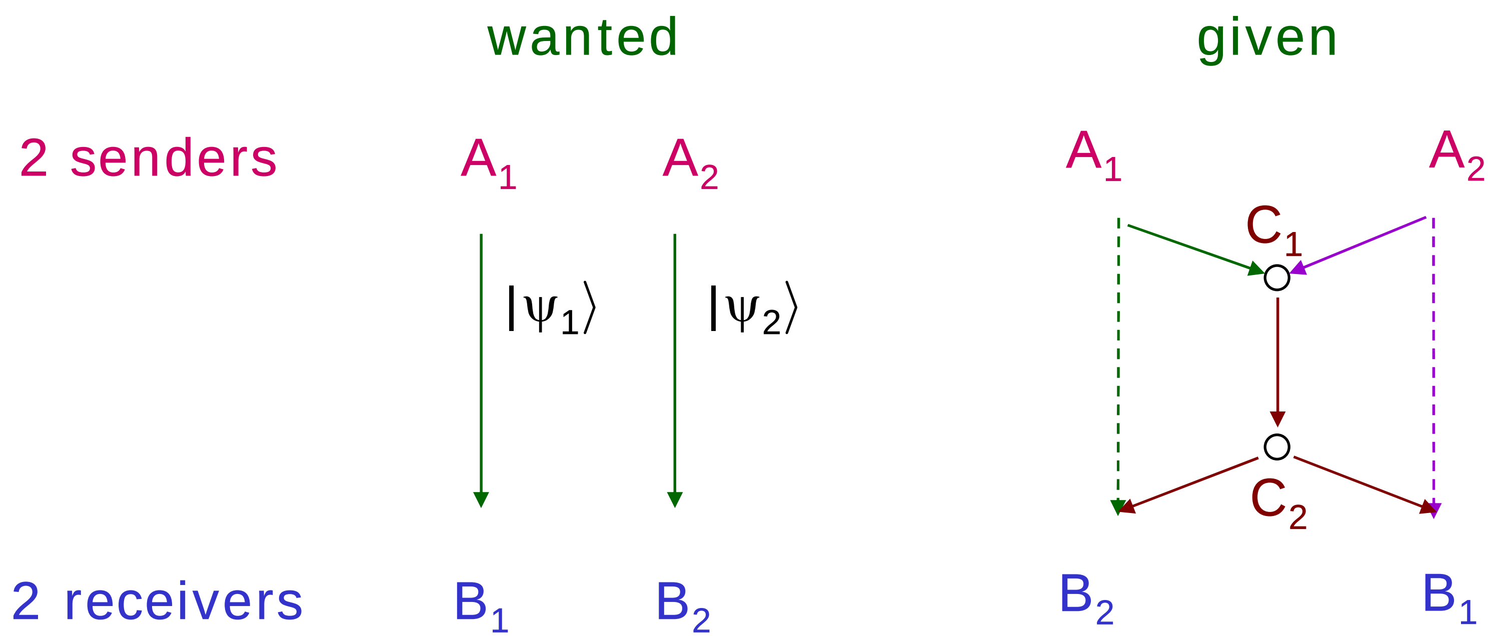
"Best" -- exact, 1-shot, & individual-rate optimal

Any question about the classical result?

Now, let's go quantum.

Motivating example : the butterfly network

Quantum: for independent $|\psi_1\rangle$, $|\psi_2\rangle$



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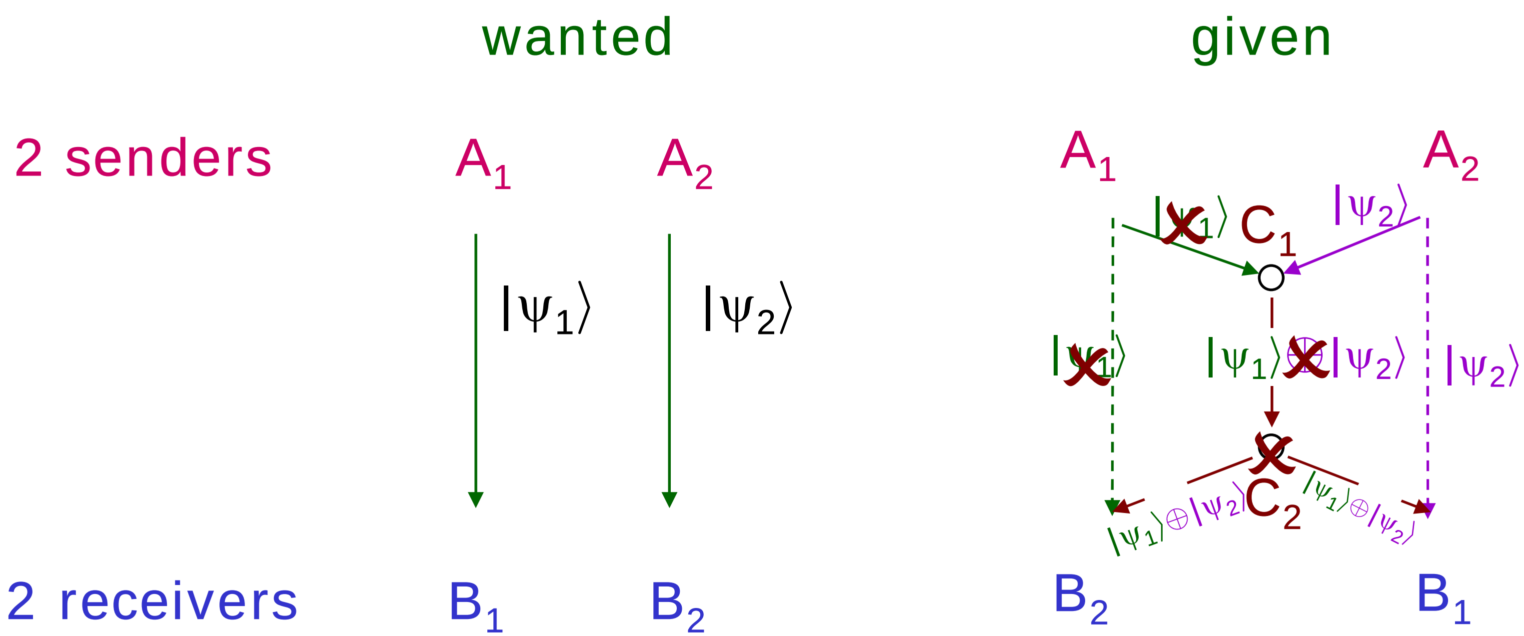
Motivating example : the butterfly network

Quantum: for independent $|\psi_1\rangle$, $|\psi_2\rangle$

awakeness-test



Assumption: "network" (all 7 channels) called as a package

Qn: how to "best" communicate from A_1 to B_1 and A_2 to B_2?

No all-optimal solution -- need to define notion of optimality

Motivating example : the butterfly network

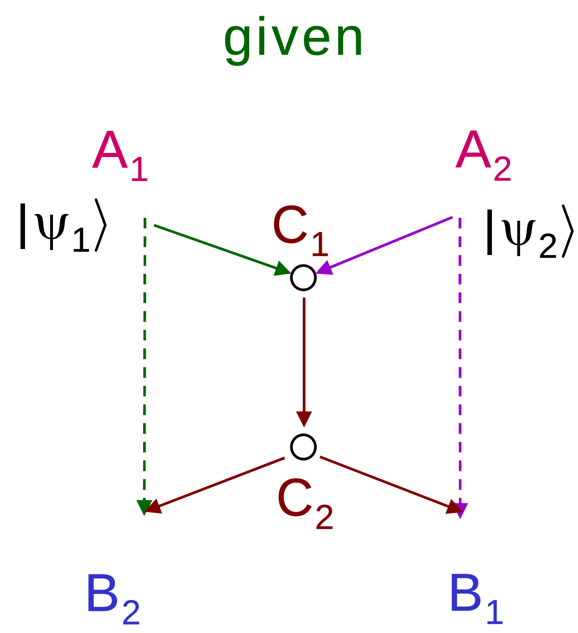
Quantum: for independent $|\psi_1\rangle$, $|\psi_2\rangle$

Prior work: [Hayashi, Iwama, Nishimura, Raymond, Yamashita 0601088] 1-use of network, fixed rate, min distortion (fidelity $\approx$ 1/2-2/3).

Here:
Asymptotic (many uses, consider info sent/network use)

Demand near-perfect transmission

Study rate trade-off between $A_1 \to B_1$ & $A_2 \to B_2$ comm

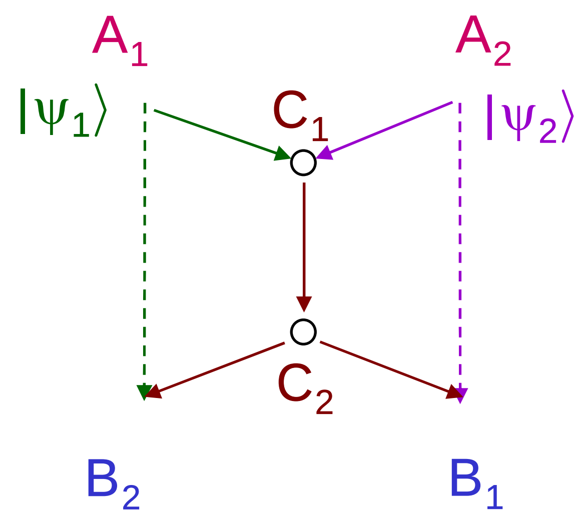


Motivating example : the butterfly network

Quantum: for independent $|\psi_1\rangle$, $|\psi_2\rangle$

Def [achievable rate region]:
If
n-uses of the network enables
n_i qubits to be faithfully sent
from A_i to B_i ,
then,
rate pair $(r_1,r_2) = (n_1/n, n_2/n)$
is "achievable."

given



Motivating example : the butterfly network

Quantum: for independent $|\psi_1\rangle$, $|\psi_2\rangle$

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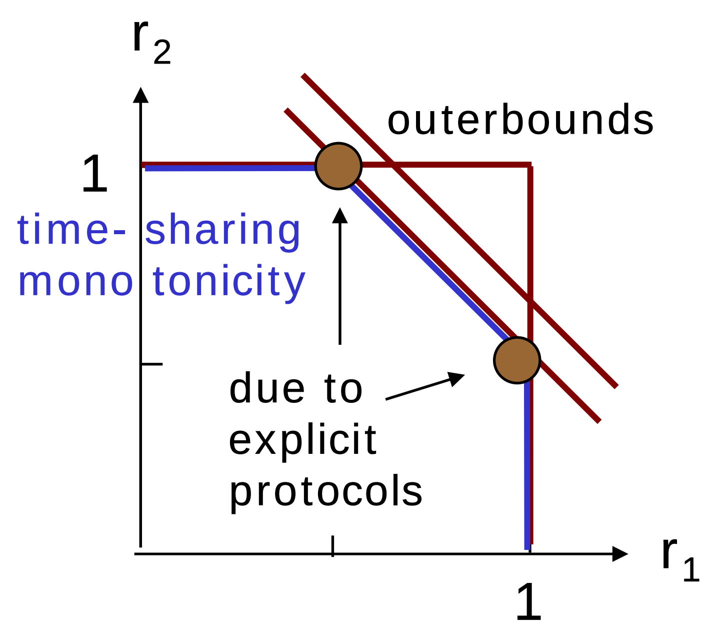


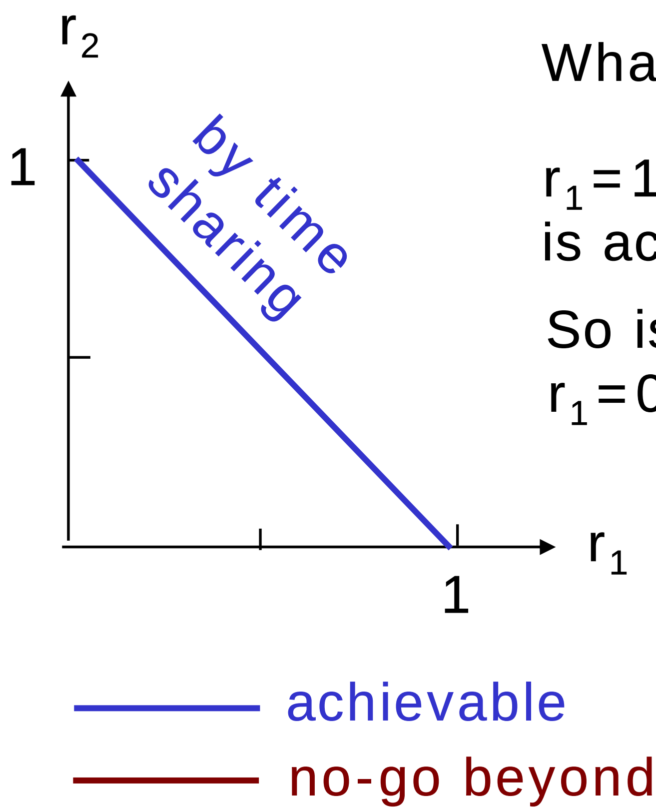
Achievable rate region is the set of all achievable rate pairs.

Goal: Find entire achievable rate region

Will consider various "assistance" (i.e., free auxiliary resources)

Motivating example : the butterfly network

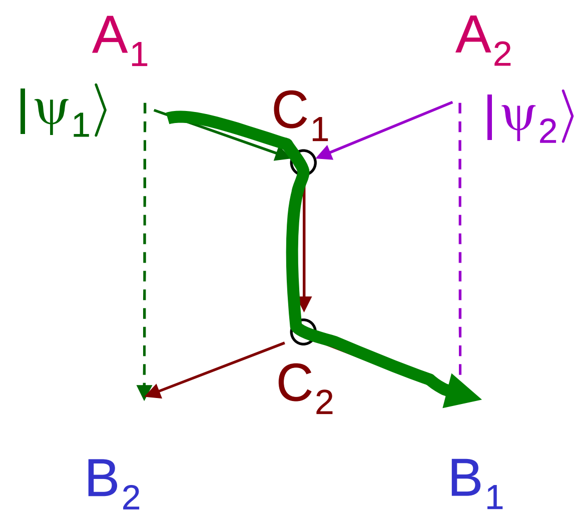
Quantum: for independent $|\psi_1\rangle$, $|\psi_2\rangle$ no assistance



What can be done

$r_1 = 1$, $r_2 = 0$
is achievable

So is
$r_1 = 0$, $r_2 = 1$



Motivating example : the butterfly network

Quantum: for independent $|\psi_1\rangle$, $|\psi_2\rangle$ no assistance

Now outerbound, **heuristic**

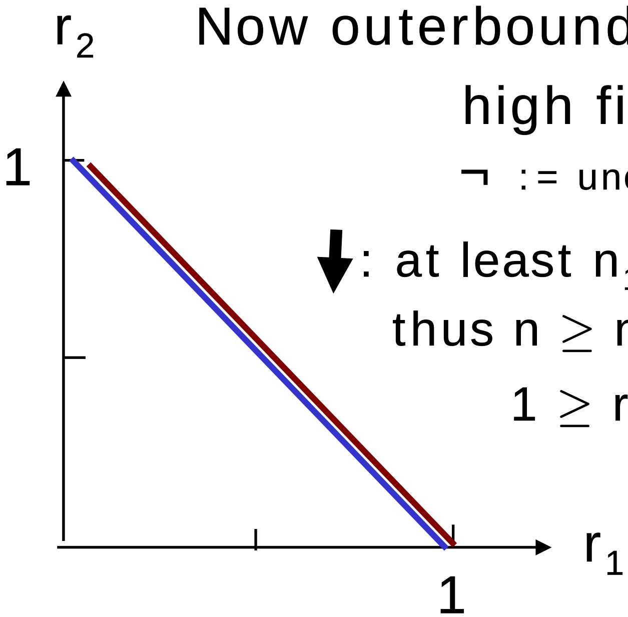
high fidelity

¬ := uncorrelated with

↓: at least n_1+n_2 qubits

thus $n \geq n_1+n_2$

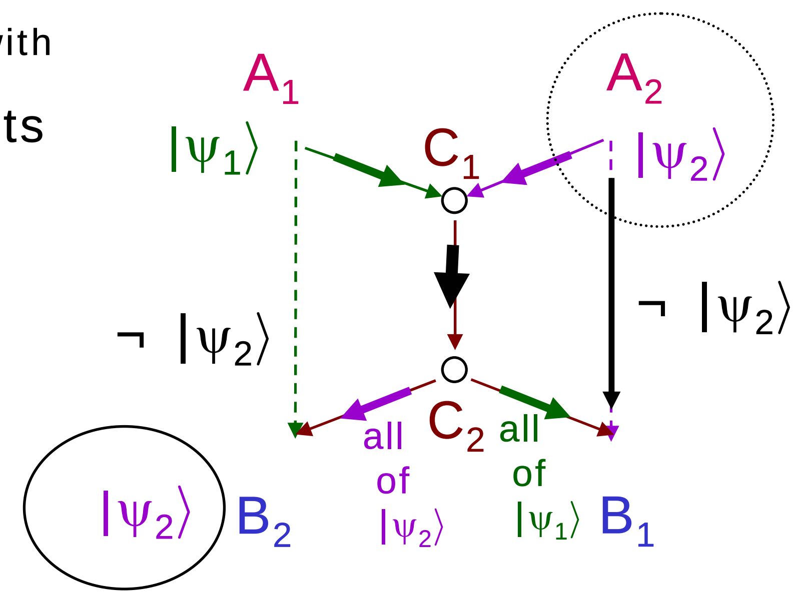
$1 \geq r_1+r_2$



achievable

no-go beyond

Asymptotic: n-qubit sent per edge (due to acyclic graph)



Optimal protocol: time sharing between 2 trivial 1-shot solutions

Motivating example : the butterfly network

Quantum: for independent $|\psi_1\rangle$, $|\psi_2\rangle$ no assistance

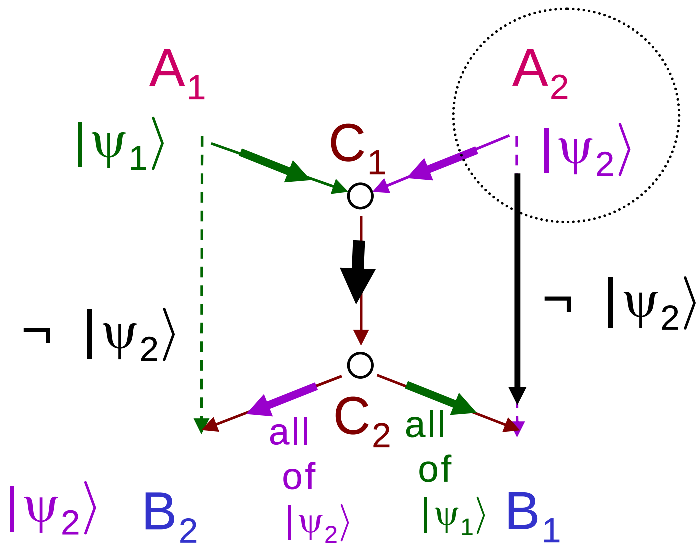
Now outerbound, **heuristic**

Asymptotic: n-qubit sent per edge (due to acyclic graph)

What's quantum? Outerbound.

Only one-copy of each quantum state ever exists.

Integrity of final received state requires undoing of correlations (not possible in this network).



Motivating example : the butterfly network

Quantum: for independent $|\psi_1\rangle$, $|\psi_2\rangle$ no assistance

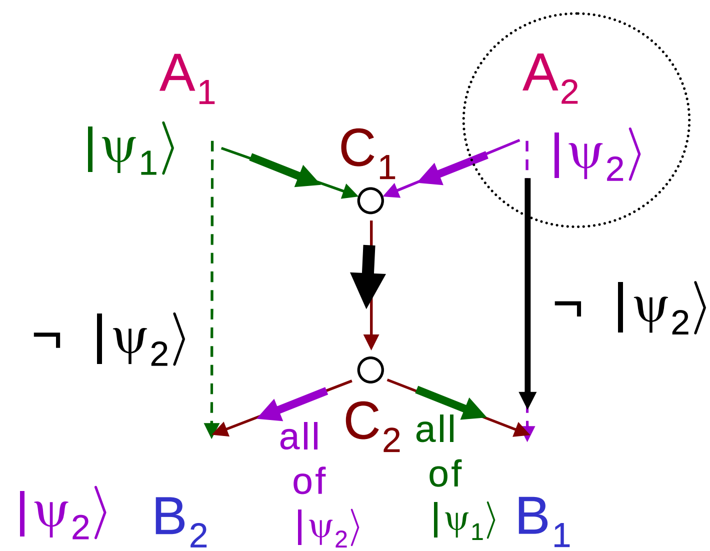
Now outerbound, **reloaded**

Asymptotic: n-qubit sent per edge (due to acyclic graph)

Think of $A_1 \to B_2$ & $C_2 \to B_2$ comm as 2 shares of $|\psi_2\rangle$ with the latter significant .

In quantum secret sharing, a significant share is at least the size of the secret.

Independence of the two messages sent from C_2 bounds the rate sum in a uniquely quantum way.



Any question about the quantum case with no assistance?

Motivating example : the butterfly network

Quantum: for independent $|\psi_1\rangle$, $|\psi_2\rangle$ free 2-way CC

What can be done

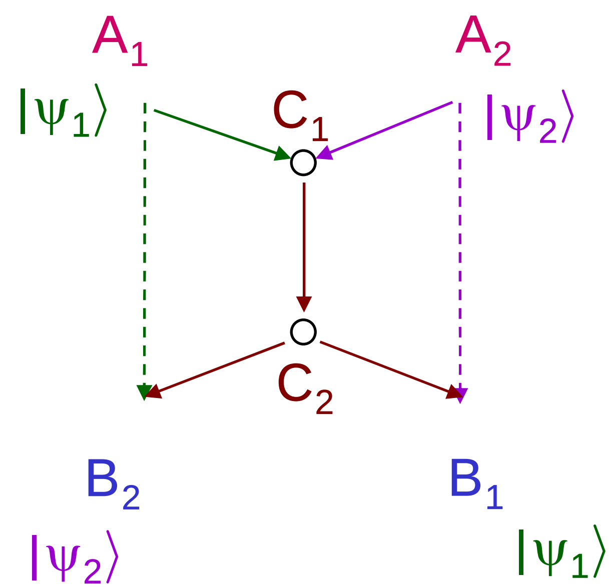
2 bits of back comm reverts a quantum channel

1 qubit forward quantum comm
+ 2 backward classical bits

gives

1 ebit of entanglement
+ 2 backward classical bits

can teleport 1 qubit backwards



Motivating example : the butterfly network

Quantum: for independent $|\psi_1\rangle$, $|\psi_2\rangle$ free 2-way CC

What can be done

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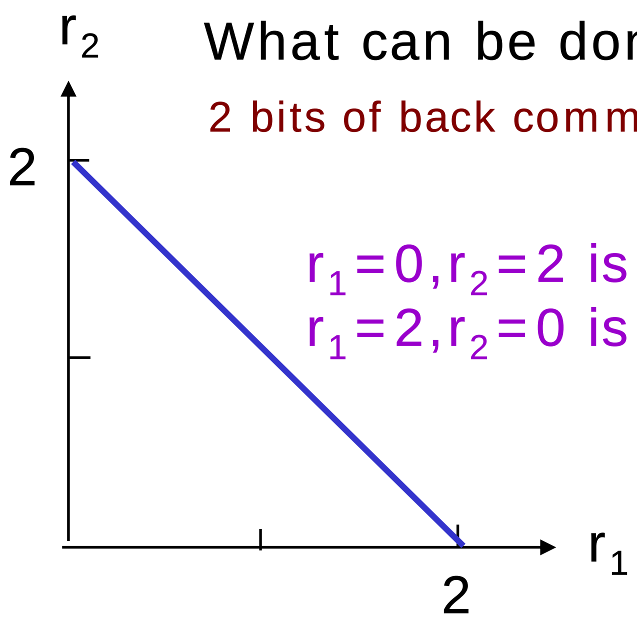
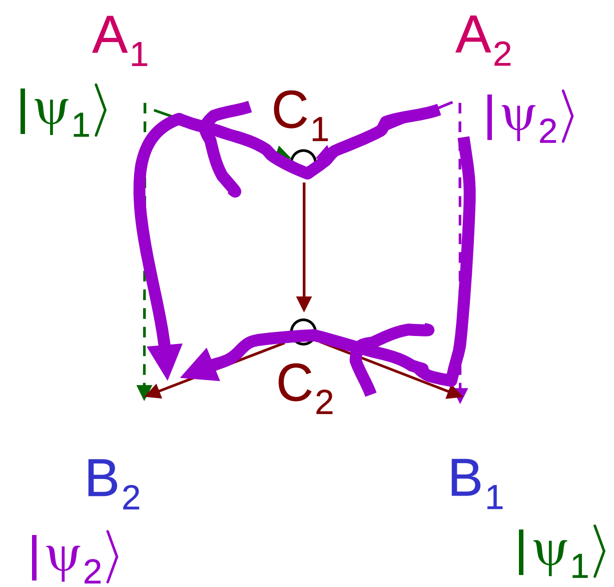


$r_1=0, r_2=2$ is achievable
$r_1=2, r_2=0$ is achievable



achievable
no-go beyond

Motivating example : the butterfly network

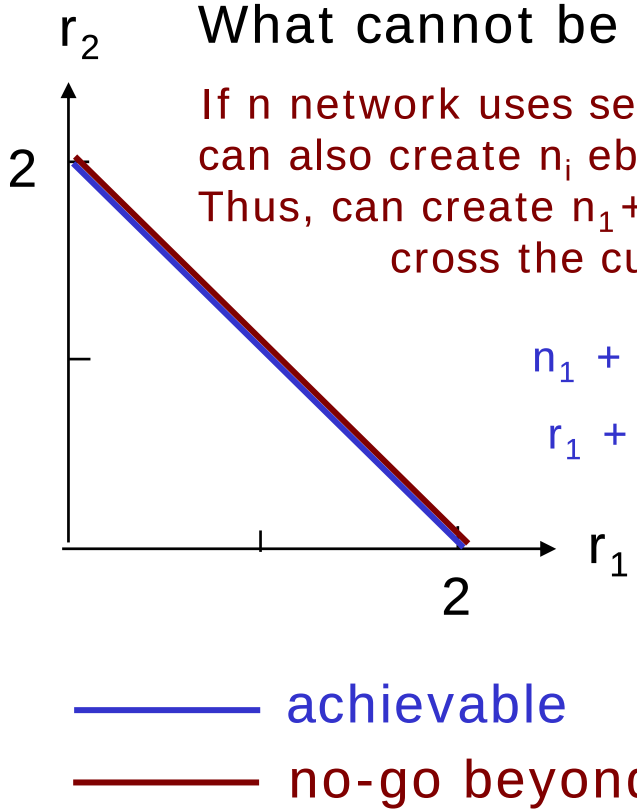
Quantum: for independent $|\psi_1\rangle$, $|\psi_2\rangle$ free 2-way CC

What cannot be done asymptotically

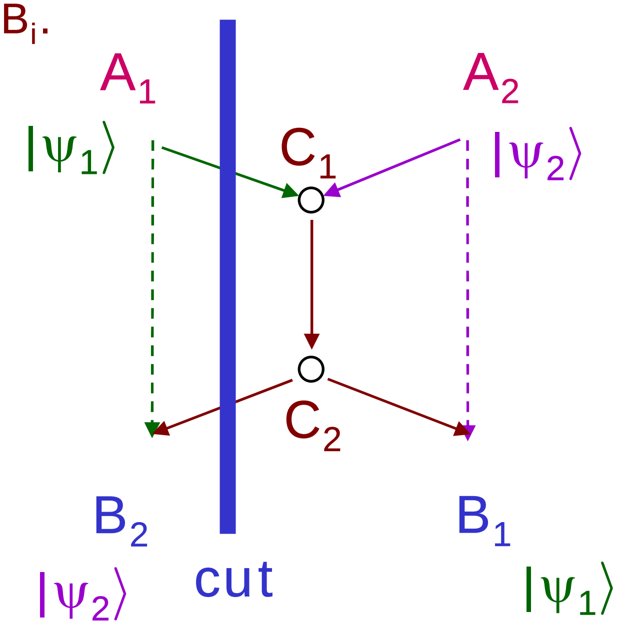
If n network uses send n_i qubits from A_i to B_i , can also create n_i ebits between A_i & B_i.
Thus, can create n_1+n_2 ebits cross the cut

$n_1 + n_2 \leq 2n$

$r_1 + r_2 \leq 2$



SAME ANALYSIS for free BACK CC



Optimal solution : time sharing of two 1-shot protocols
Note: free two-way CC no better than 4 bits of back comm

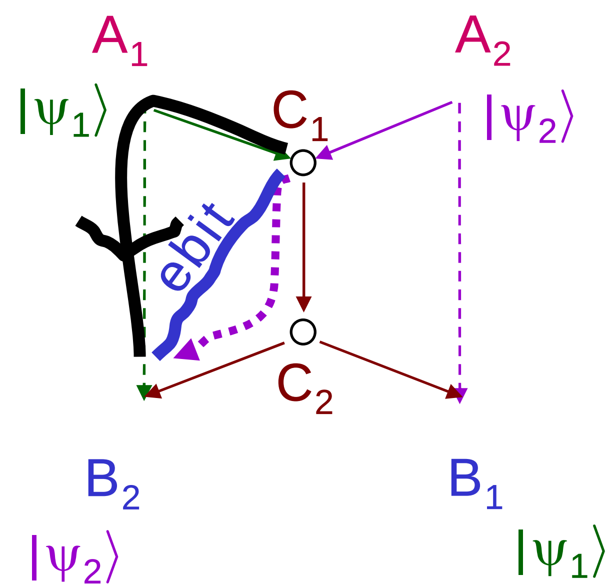
Any question about the quantum case with
free 2-way or backward
classical communication assistance?

Motivating example : the butterfly network

Quantum: for independent $|\psi_1\rangle$, $|\psi_2\rangle$ free FORWARD CC

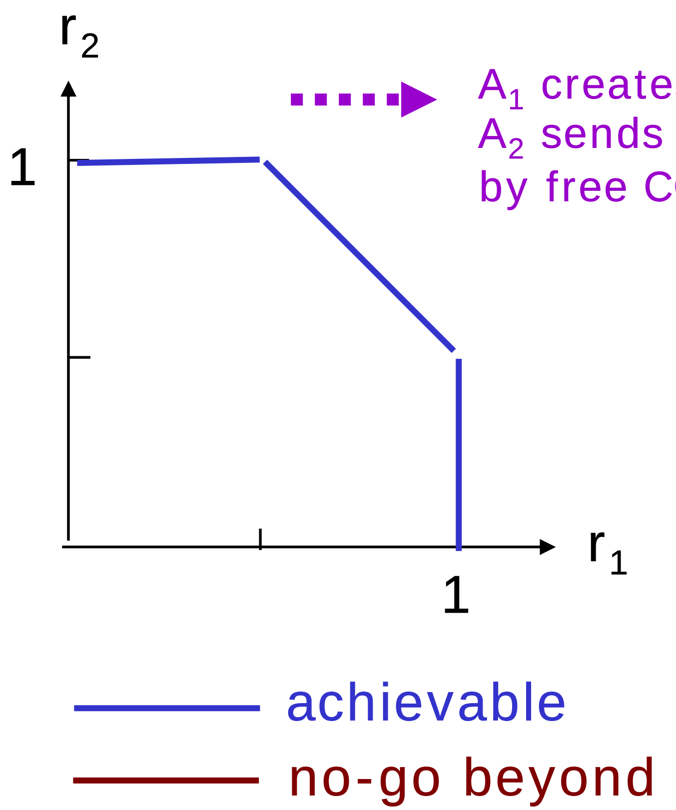
In this scenario, some but not all channels can be reversed.

A_1 creates an ebit between C_1 and B_2
A_2 sends $|\psi_2\rangle$ to C_1 who teleports it to B_2 by free CC to C_2 to B_2

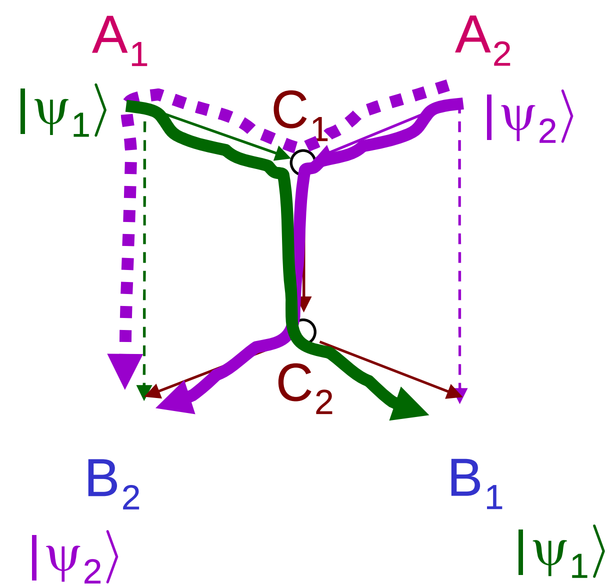


Motivating example : the butterfly network

Quantum: for independent $|\psi_1\rangle$, $|\psi_2\rangle$ free FORWARD CC



A1 creates an ebit between C1 and B2
A2 sends $|\psi_2\rangle$ to C1 who teleports it to B2
by free CC to C2 to B2



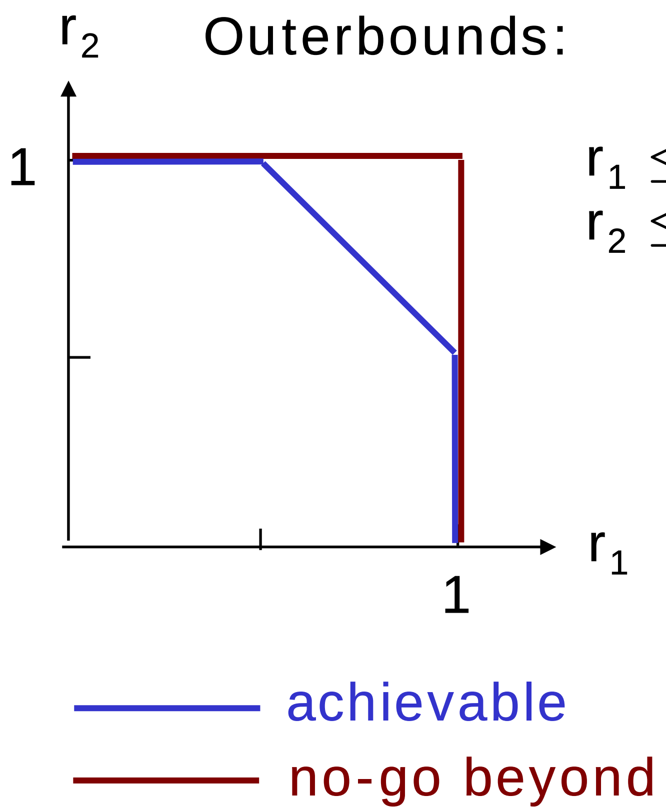
$n_1=1$, $n_2=2$ for $n=2$, hence $(r_1, r_2) = (0.5,1)$ achievable
So is (1,0.5) by symmetry. Time sharing the two.

Motivating example : the butterfly network

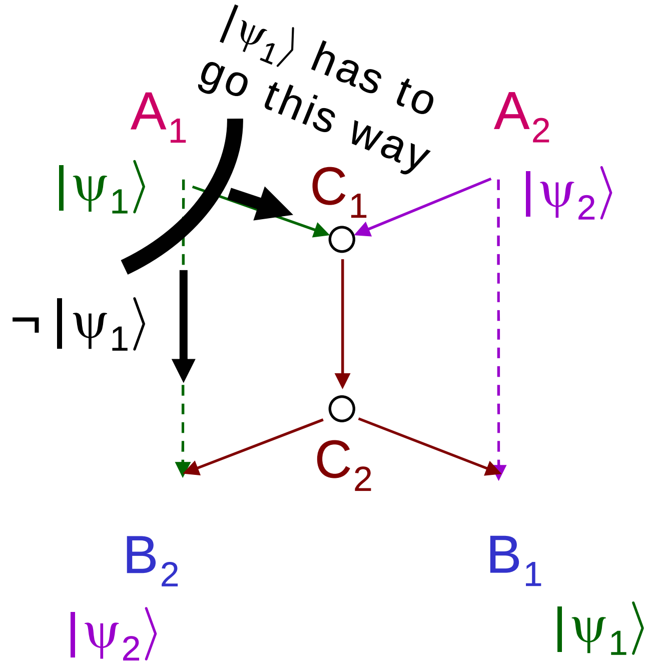
Quantum: for independent $|\psi_1\rangle$, $|\psi_2\rangle$ free FORWARD CC

Outerbounds:

$r_1 \leq 1$
$r_2 \leq 1$



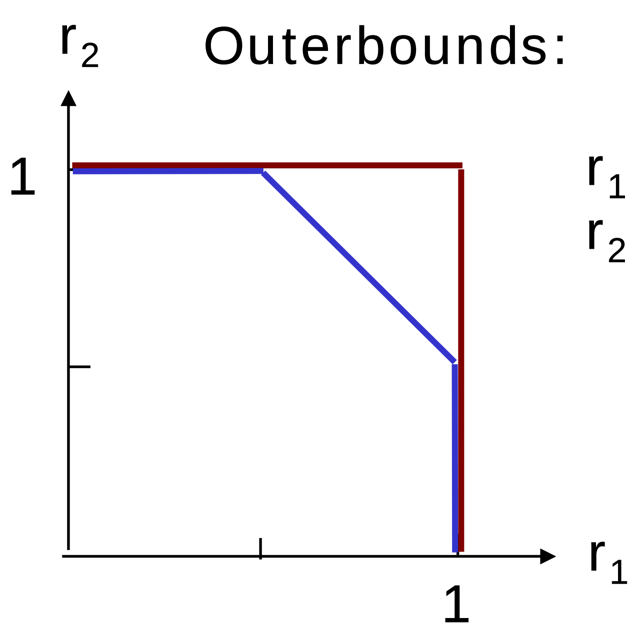
Heuristic



Motivating example : the butterfly network

Quantum: for independent $|\psi_1\rangle$, $|\psi_2\rangle$ free FORWARD CC

Outerbounds:



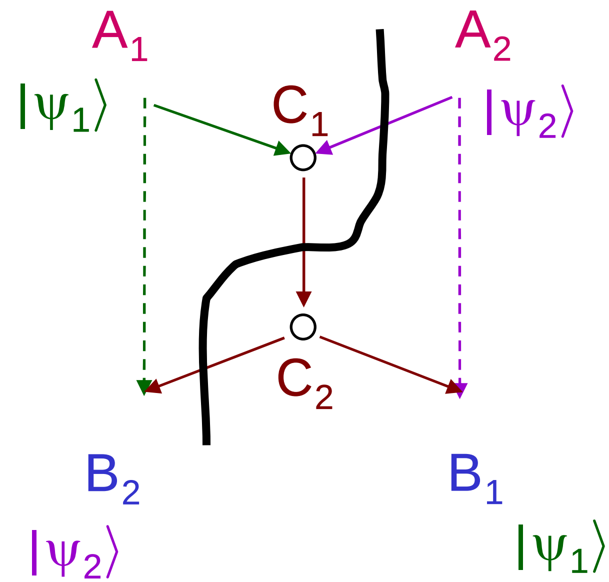
$r_1 \leq 1$

$r_2 \leq 1$

achievable

no-go beyond

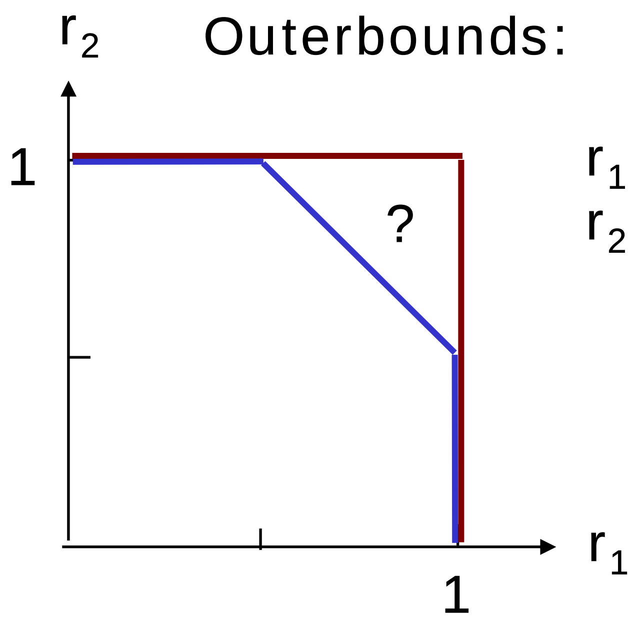
More rigorously:



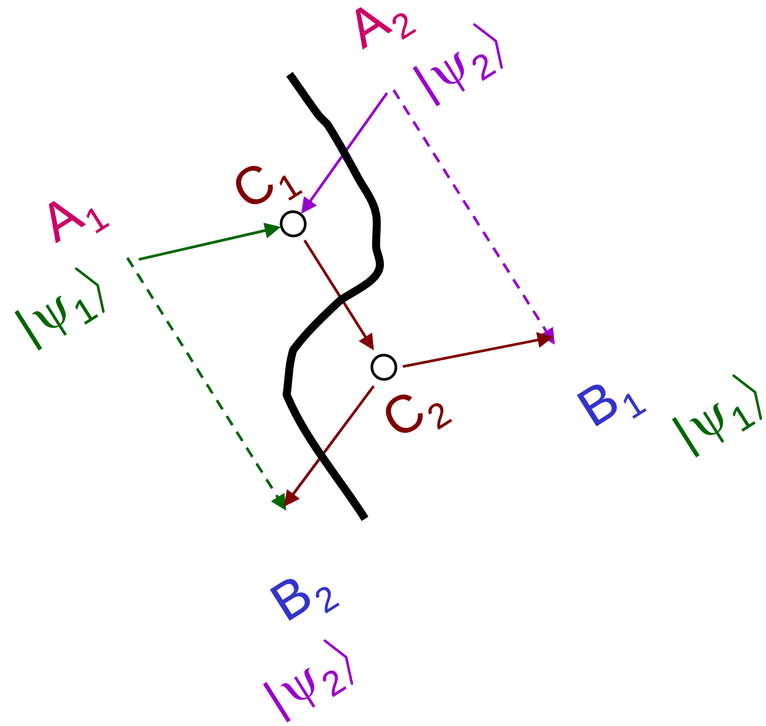
Motivating example : the butterfly network

Quantum: for independent $|\psi_1\rangle$, $|\psi_2\rangle$ free FORWARD CC

Outerbounds:



$r_1 \leq 1$
$r_2 \leq 1$



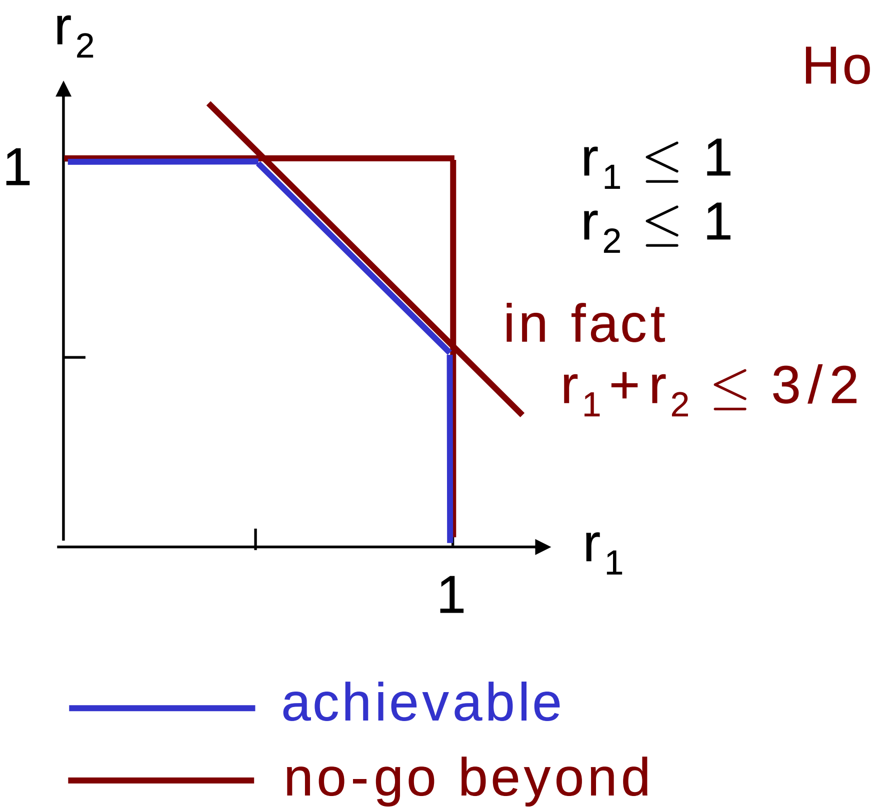
achievable

no-go beyond

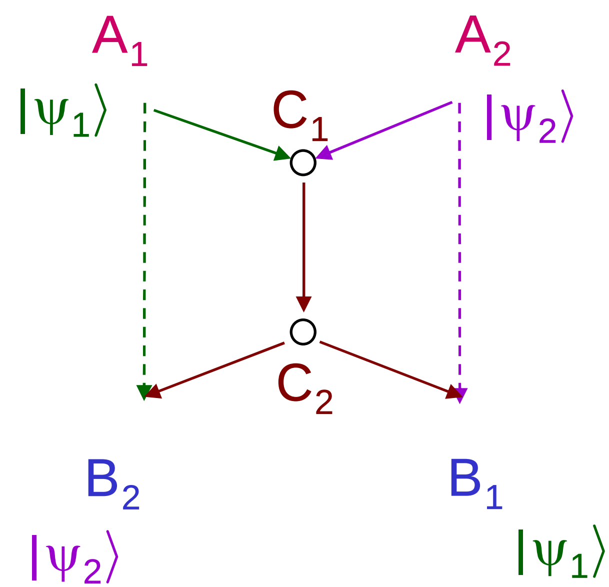
If $|\psi_1\rangle$ has n_1 qubits, n_1 qubits has to be sent from L $\rightarrow$ R (even with unlimited backward qubit comm). Also, q/c encoding of q state does not reduce the quantum part.

Motivating example : the butterfly network

Quantum: for independent $|\psi_1\rangle$, $|\psi_2\rangle$ free FORWARD CC



How many ways can A_2 send to B_2?



Motivating example : the butterfly network

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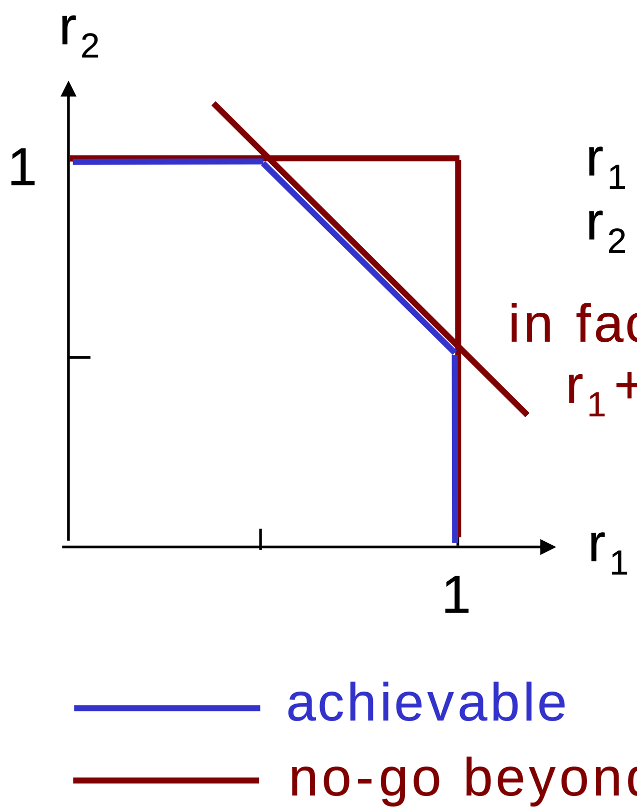


$r_1 \leq 1$
$r_2 \leq 1$

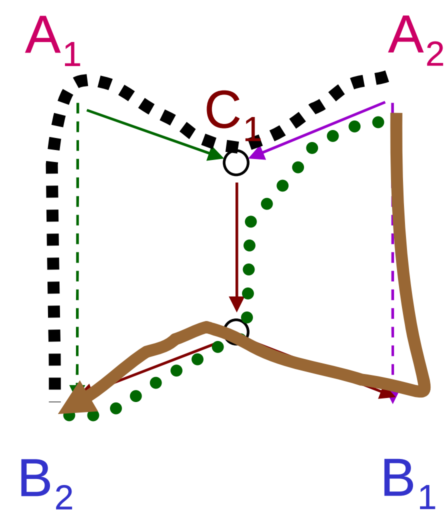
in fact

$r_1 + r_2 \leq 3/2$

achievable

no-go beyond

How many ways can A_2 send to B_2?

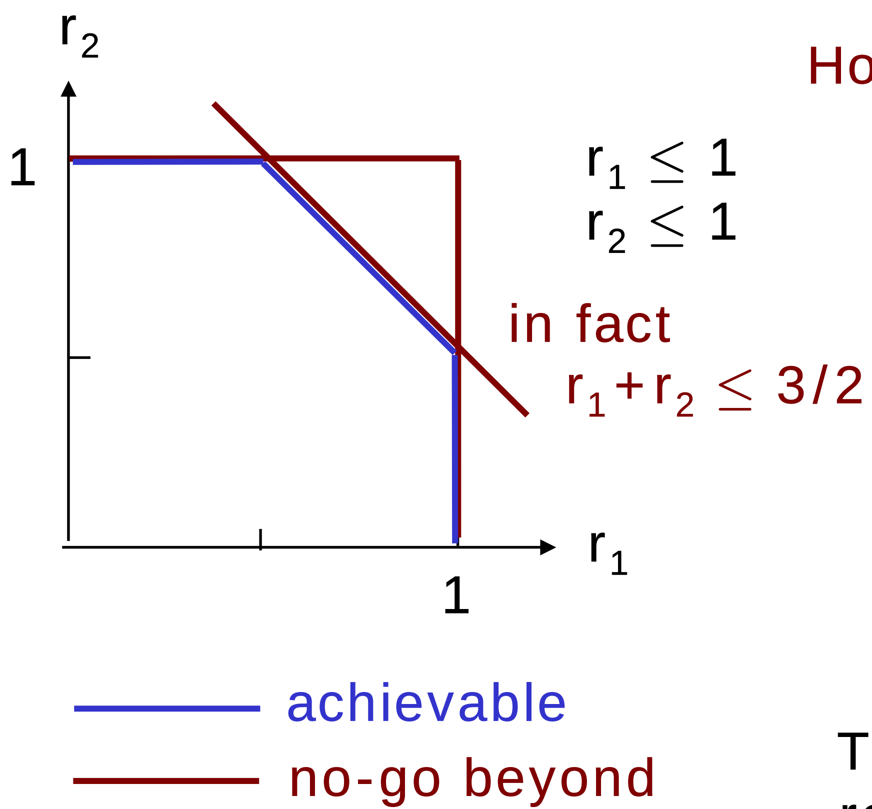


Path via B_1 (a sink) is invalid!

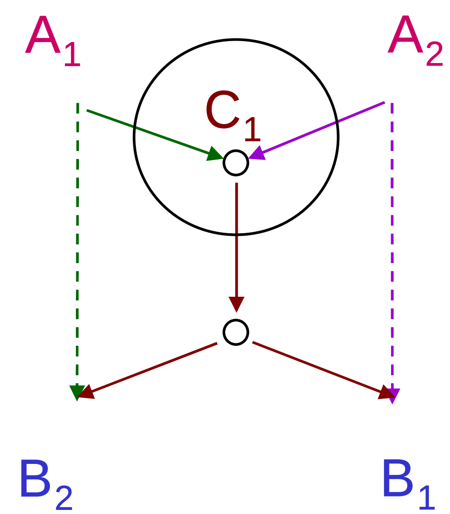
i.e. all comm from A_2 to B_2 goes through C_1. Similarly for A_1 to B_1

Motivating example : the butterfly network

Quantum: for independent $|\psi_1\rangle$, $|\psi_2\rangle$ free FORWARD CC



How many ways can A_2 send to B_2?

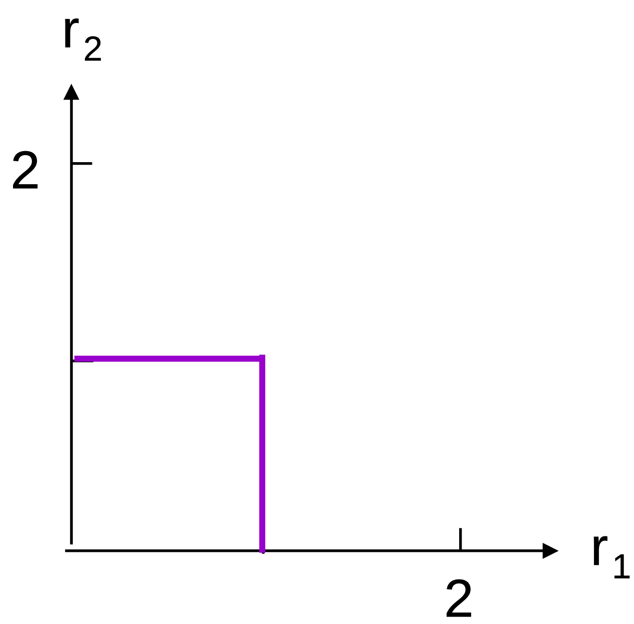


To send a qubit from A_i to B_i require 2 edges going in/out of C_1. Thus, if $n_1 + n_2$ qubits are sent using n networks, $n_1 + n_2 \leq 3n/2$

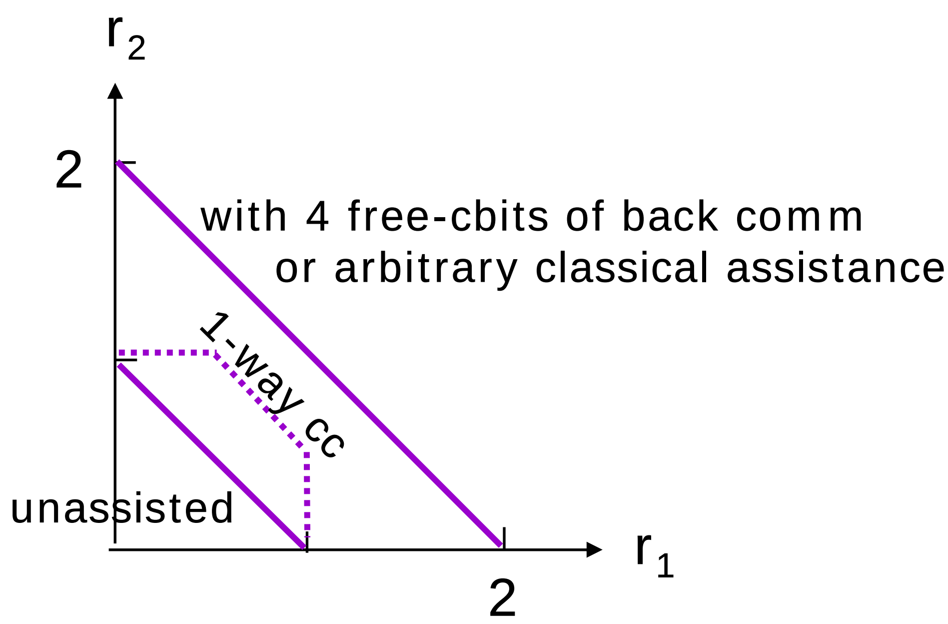
Any question about the quantum case with free forward classical communication assistance?

Summary for the butterfly network:

Uniquely optimal
classical rate region



Rate optimal - high fidelity
quantum rate regions



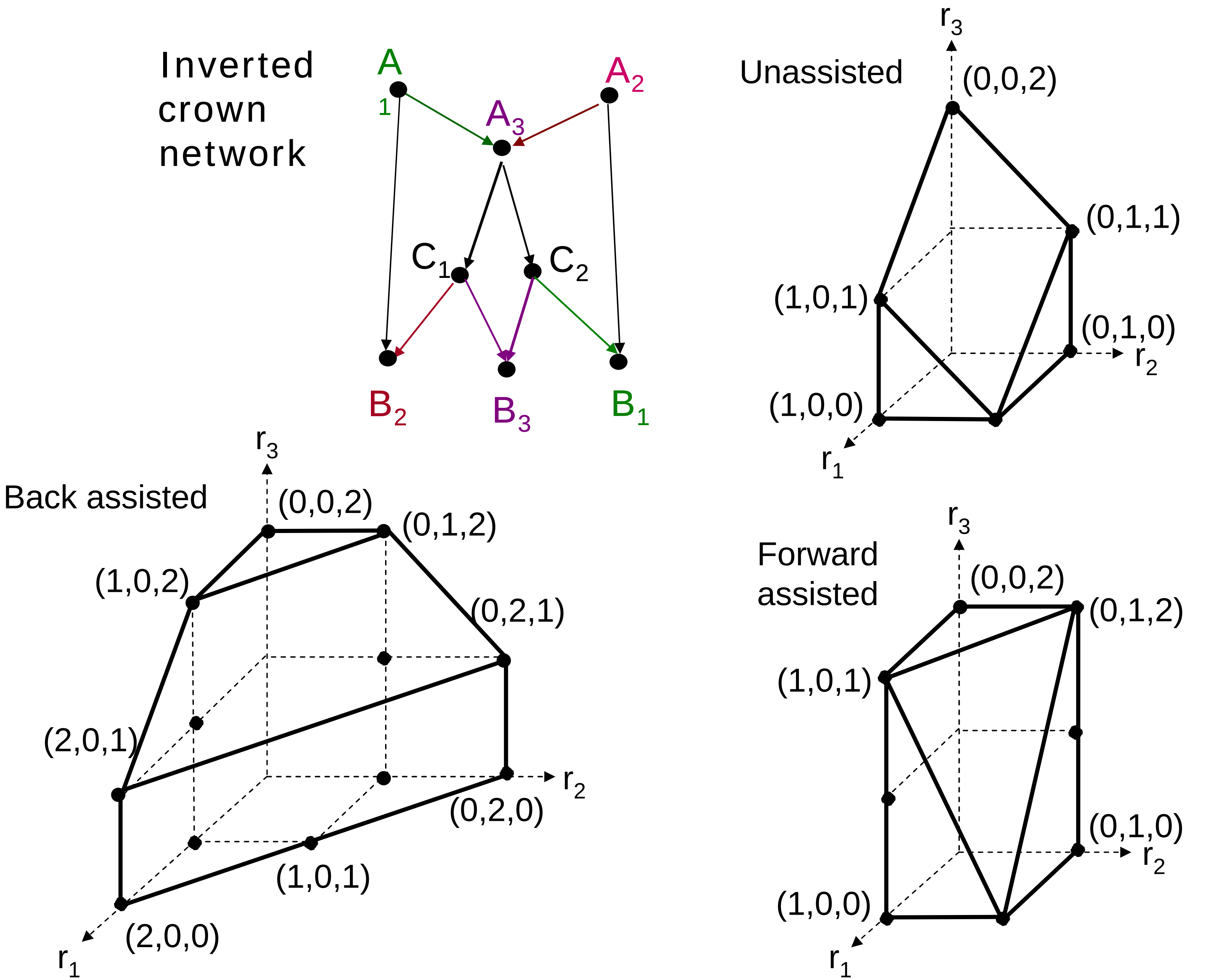
Summary for the butterfly network:

Quantum case:

The surprise is the simpicity (boringness) of the optimal solutions : each "leg" is either used for $A_1 \rightarrow B_1$ comm or $A_2 \rightarrow B_2$ comm

Contrast to the classical case, quantum info runs down a network like incompressible water running down pipes.

Does this simplifying feature hold in more general networks?



Inverted crown network
A1
A2
A3
C1
C2
B2
B3
B1
Unassisted
r3
r2
r1
(0,0,2)
(0,1,1)
(1,0,1)
(0,1,0)
(1,0,0)
Back assisted
r3
r2
r1
(0,0,2)
(0,1,2)
(1,0,2)
(0,2,1)
(2,0,1)
(0,2,0)
(1,0,1)
(2,0,0)
Forward assisted
r3
r2
r1
(0,0,2)
(0,1,2)
(1,0,1)
(0,1,0)
(1,0,0)

Generalization (possible directions):

- 2-pair $\rightarrow$ k-pair comm problem
  i.e. communication of independent messages between
  specific sender-receiver pairs

- arbitrary directed acyclic graph
  - vertices represent players
  - arbitrary number of "helpers" (Ci's)
  - weighted edges for noiseless channels
    of weighted capacity in 1 network call

Task: .

Goal: find asymptotic achievable rate region

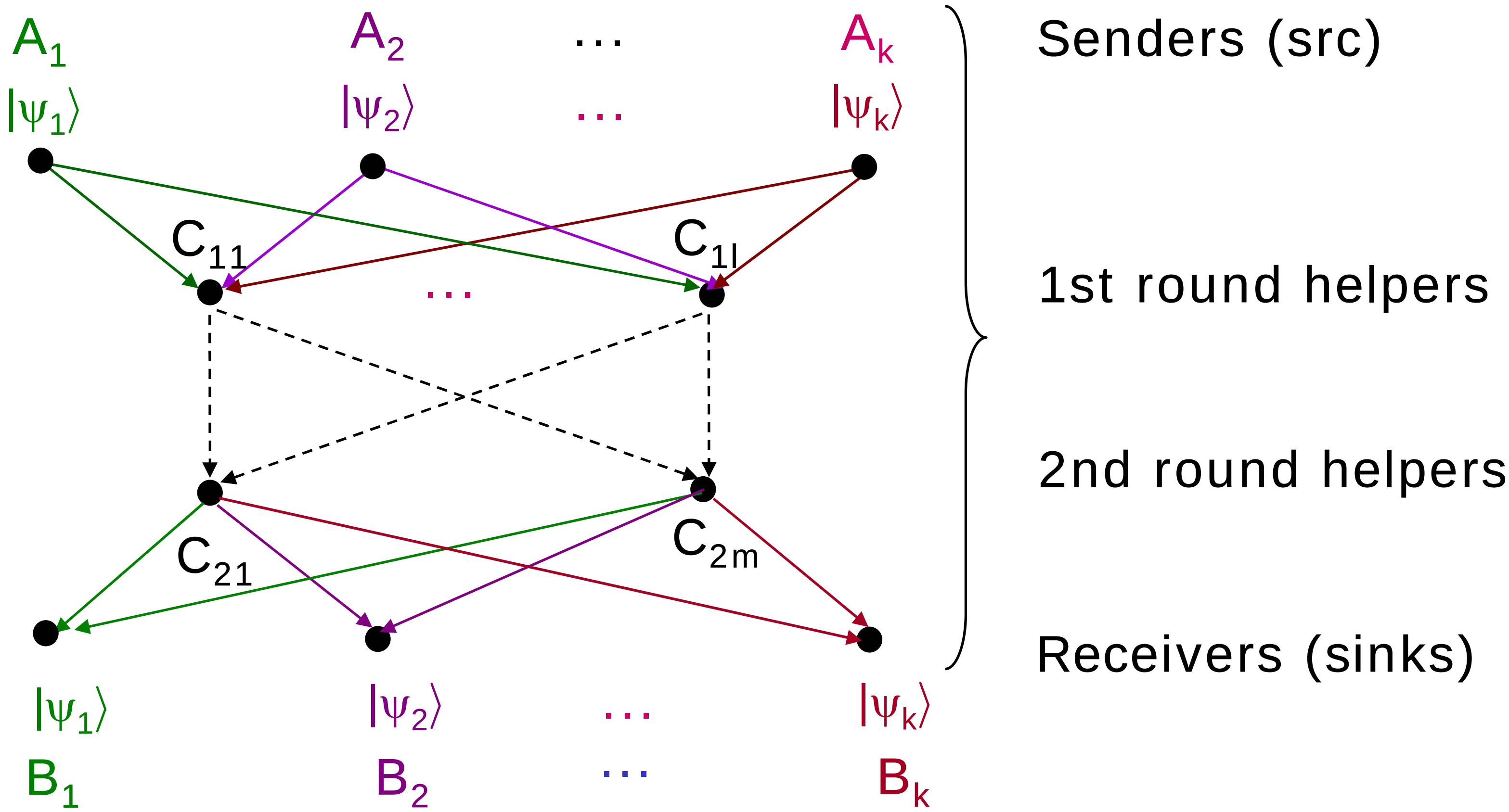
Generalization (results):

- Optimality of "rerouting" holds for

  (a) all possible scenarios for "shallow" networks

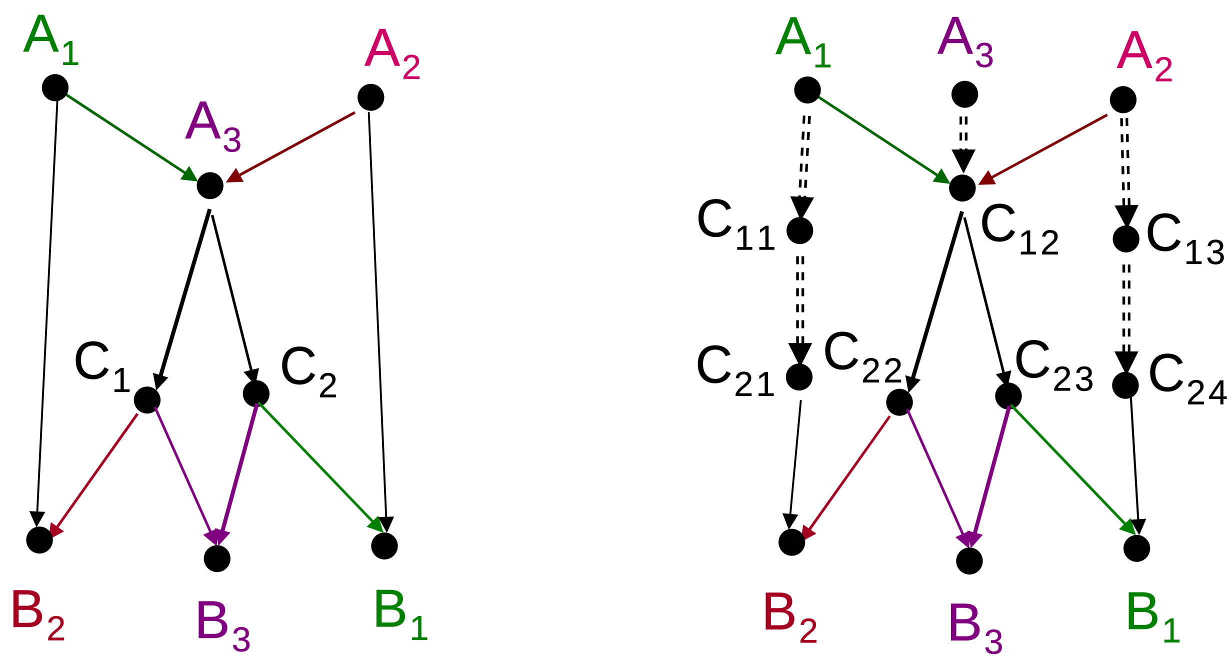
  (b) for back comm assisted, 2-pair comm problem

Shallow networks:



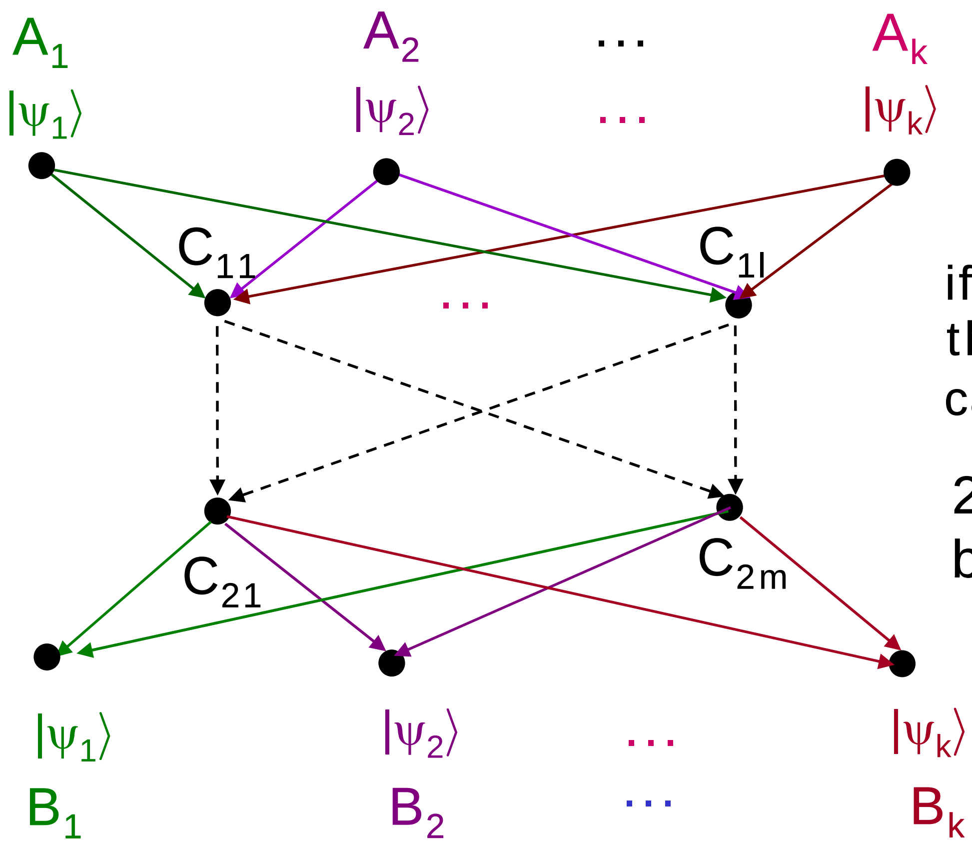
These sets of players: senders, 1st & 2nd round helpers and receivers can overlap. We assign multiple vertices to each common party, linked by channels with unlimited capacities.

Shallow networks (examples):



links with unlimited capacities

Shallow networks:



sender's transmission must be uncorrelated

if 2st round helpers entangle their shares from diff $|\psi_i\rangle$'s, can't get 2nd round correct

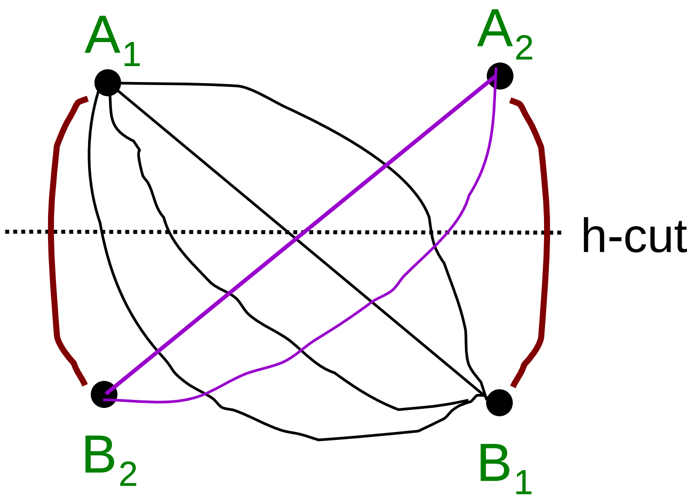
2nd round helpers must be disentangling them

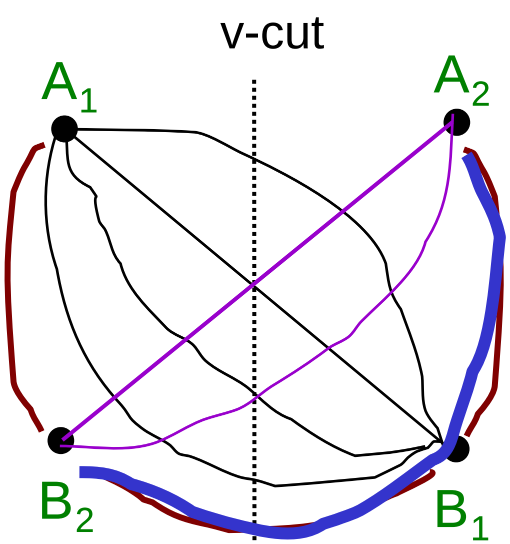
transmissions to different B_i's are uncorrelated

Argument applies to scenarios of CC assistance, but more paths have to be considered.

2-pair back-assisted networks:

undirected





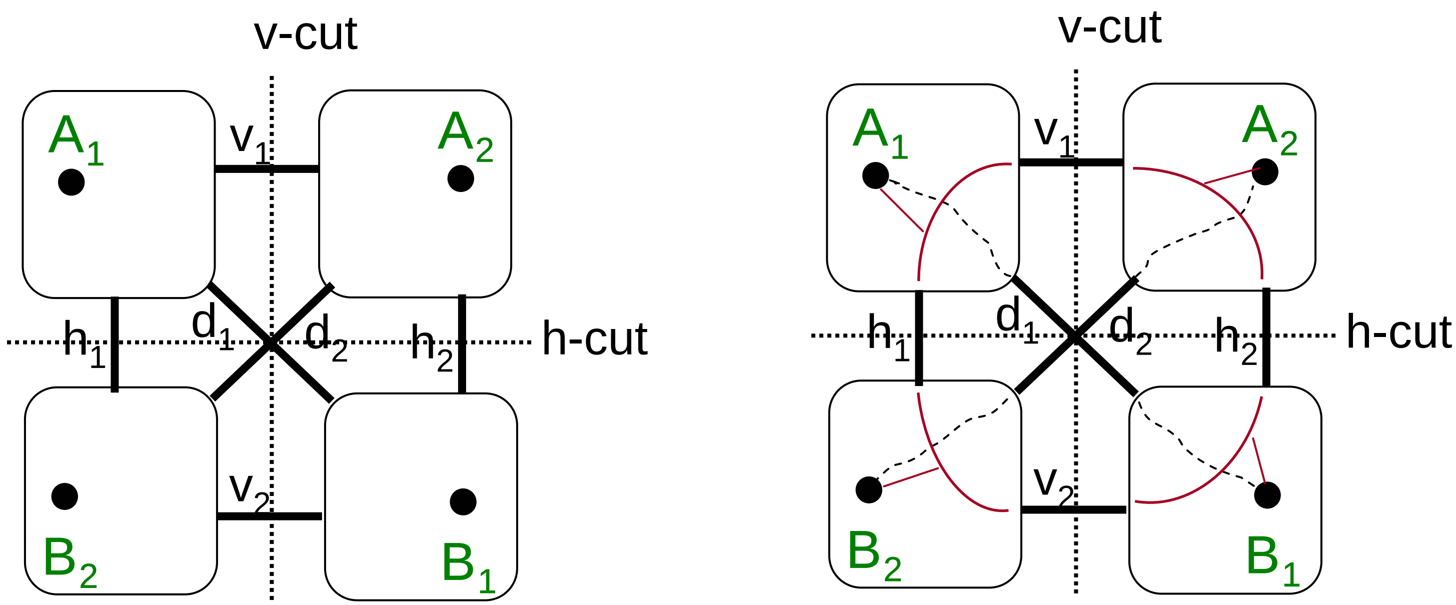
The rate sum is upper bounded by the min of the two mincuts

The diagonal ones are always "correct"

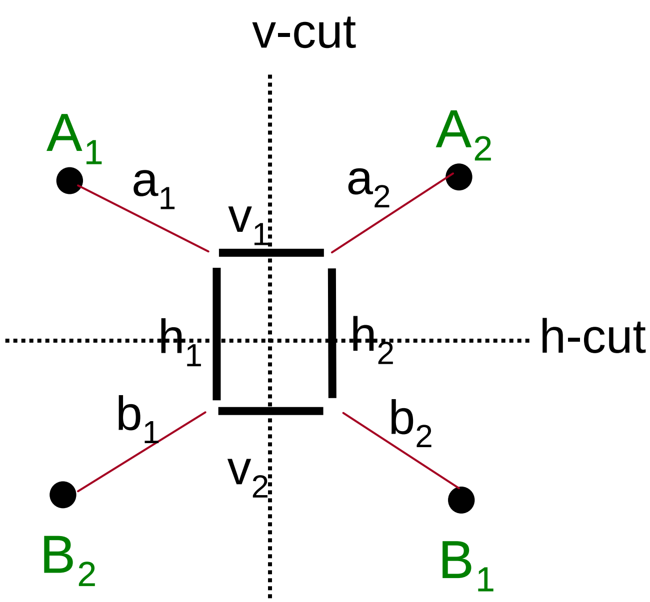
The vertical and horizontal ones need to pair up to contribute

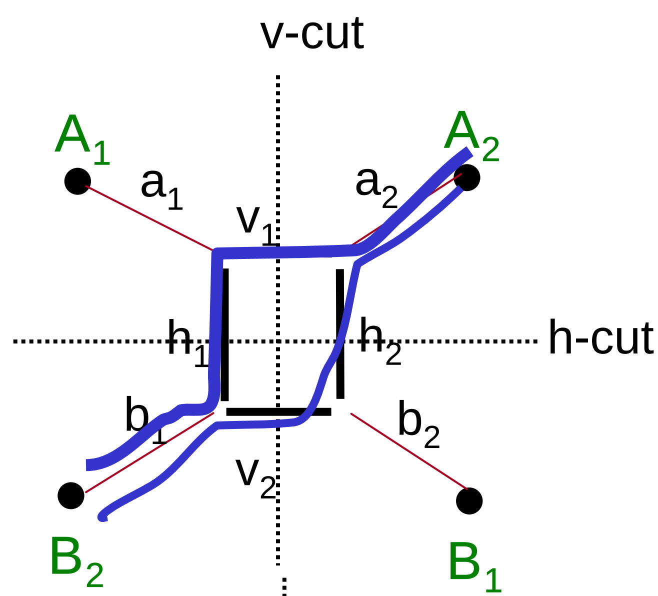
The min of the v- & h-mincuts is an achievable rate sum

2-pair back-assisted networks:



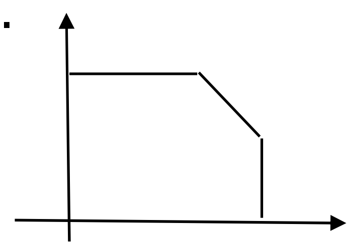
2-pair back-assisted networks:

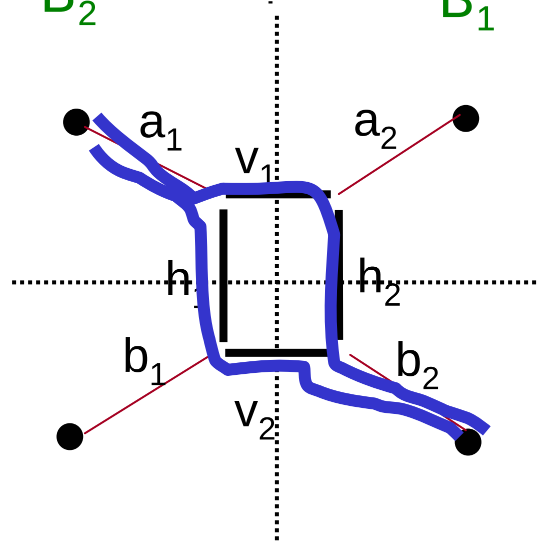




for r_2

Trading between these generates all possible rate pairs.





for r_1

Open problems:

? Optimality for more general cases ??

? Cyclic graphs ??

? Can we adapt some of the father/mother type of results ??

Entanglement assisted case:

Inner bound = classical region
Is it matched by the outerbound?