Mathematical Analysis of Throughput Bounds in Random Access with ZIGZAG Decoding

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***Abstract*—**
We investigate the throughput improvement that ZIGZAG decoding (Gollakota and Katabi (2008)) can achieve in multi-user random access systems. ZIGZAG is a recently proposed 802.11 receiver design that allows successful reception of packets despite collision. Thus, the maximum achievable throughput of a wireless LAN can be significantly improved by using ZIGZAG decoding. We analyze the throughput bounds in three different idealized slotted multi-access system models for the case when ZIGZAG decoding is used. We also provide results for the Aloha and CSMA models where exact closed form solutions are infeasible to calculate. Our analysis and simulation results show that ZIGZAG decoding can significantly improve the maximum throughput of the random access system.

I. INTRODUCTION

In this paper, we investigate how much throughput improvement ZIGZAG decoding [1] can achieve. ZIGZAG is a recently proposed 802.11 receiver design that combats hidden terminals. In ZIGZAG, the receiver can decode two consecutive signals of two colliding packets and successfully receive both packets despite collision. In other words, if the same two packets collide twice (with a small bit offset difference), the receiver can receive both of those packets. Thus, the maximum achievable throughput of a wireless LAN can be significantly improved by using ZIGZAG decoding.

We look at three different idealized slotted multi-access system models to investigate the performance benefits of ZIGZAG decoding; 1) N-user slotted random access system, 2) stabilized slotted Aloha, and 3) slotted CSMA with mini-slots. Prior work has already analyzed these simple systems and has given throughput bounds (see, for example, [2], [3] for summaries and reviews of prior work). We extend these well known results to the case when ZIGZAG decoding is used. We also provide results for the Aloha [4] and CSMA [5] models where exact closed form solutions are infeasible to calculate. We concentrate on simple random access protocols and do not consider conflict resolution techniques such as [6], [7] or capture effects (e.g. [8], [9]) for the simplicity of analysis. Related recent work in [10] considers random Aloha-type protocols in networks with "soft collisions" and multi-packet reception capabilities (such as CDMA systems). The ZIGZAG decoding feature that we consider in the present paper can be viewed as a form of multi-packet reception, but is structurally quite different from the work in [10] and it does not require a CDMA structure.

Most of the classical assumptions for idealized slotted multi-access models used in prior work also hold in our work unless stated otherwise. For example, our models have slotted time with slot boundaries at integer times ($t \in \{0, 1, 2, ...\}$), packets have fixed sizes and their transmission time equals exactly one slot, and if just one node sends a packet in a given slot, the packet is correctly received at the receiver. Also, we assume that each packet involved in a collision must be retransmitted later until the packet is successfully received. A node with a packet that must be retransmitted is said to be backlogged.

However, to apply ZIGZAG into our analysis, we redefine the term *'Collision'* and make the following simplified assumptions: Now, 'Collision' occurs on a slot when 3 or more users attempt transmission in a given time slot. In this case, no packets are delivered to the receiver. If exactly 2 users transmit packets on a slot, we say that this is a 'ZIGZAG' case which is decodable using ZIGZAG decoding. Either an 'Idle,' 'Success,' 'ZIGZAG,' or 'Collision' event happens on every slot, (corresponding to whether 0, 1, 2, or more than 2 packets were transmitted in that slot, respectively) and this feedback is explicitly given ('0,' '1,' 'ZIGZAG,' 'C') to all users at the end of each slot. If a 'ZIGZAG' event occurs, that slot is automatically extended into 2 slots. The two colliding users know that they have collided in a ZIGZAG event, and always retransmit the same packet in the next slot. Other users also know about this and thus remain silent in the next slot. If a 'ZIGZAG' event occurs, exactly 2 packets can be perfectly received at the receiver during 2 time slots. Hence the average throughput during this period is 1 packet/slot. Finally, we ignore other aspects of ZIGZAG decoding such as decoding failure or 3 packet decoding.[1]

While we use a slotted model here, it is important to recall that ZIGZAG decoding relies on packet transmissions to arrive

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[1] In a real system, there could be cases where ZIGZAG decoding may fail for some practical reasons. Also, there could be cases where 3 or more colliding packets are decodable. The ZIGZAG paper [1] does describe these cases but we ignore them for simplicity of analysis.

Fig. 1: A timeline showing the various kinds of frames ‘0’, ‘1’, ‘C’, ‘ZIGZAG’

at the receiver so that the first bits of each packets are non-aligned. This is consistent with a slotted model if slot sizes are several bits larger than packet sizes, and if transmissions are randomized with respect to the initial bit alignment. A general model allows for some probability p_f for a ZIGZAG frame failure due to accidental perfect alignment. However, in this paper, we treat the ideal case of $p_f = 0$ for simplicity.

II. Slotted Random Access

We first look at a simple idealized random access protocol for a N-user system where each user has an infinite amount of data to send. In this protocol, we use ZIGZAG decoding as follows. The timeline is decomposed into *frames* of size either one or two slots, and these frames define renewal events (see Fig. 1). At the beginning of a frame, all N users independently transmit a packet with probability q ($0 \leq q \leq 1$), and exactly one of the following four events happen; 1) *Idle*: nobody transmits any packet, 2) *Success*: exactly 1 user transmits a packet, 3) ZIGZAG: exactly 2 users transmit a packet, or 4) *Collision*: 3 or more users attempt transmission. Then the receiver gives one of the 4 following feedback messages at the end of the first slot of the frame:

$$Feedback = \begin{cases} \text{‘0’} & \text{if idle (i.e., no packets attempted transmission).} \\ \text{‘1’} & \text{if success (i.e., exactly 1 packet attempted transmission).} \\ \text{‘ZIGZAG’} & \text{if exactly 2 packets attempted transmission.} \\ \text{‘C’} & \text{if 3 or more packets attempted transmission.} \end{cases}$$

The frame has size 1 slot if the feedback is ‘0,’ ‘1,’ or ‘C.’ If the feedback is ‘ZIGZAG,’ then the frame has a size of 2 slots. Recall that in the second slot of a ‘ZIGZAG’ frame, the same two users transmit their packets again while all other users remain silent. Thus, in a ‘ZIGZAG’ frame, exactly two packets are successfully transmitted by the end of the 2-slot frame.

In this system, the probability of ‘1’ (success) and ‘ZIGZAG’ is given as follows:

$$P_1 = \binom{N}{1} q(1-q)^{N-1} = Nq(1-q)^{N-1} \tag{1}$$

$$P_{\text{zigzag}} = \binom{N}{2} q^2(1-q)^{N-2} = \frac{N(N-1)}{2} q^2(1-q)^{N-2} \tag{2}$$

Also, the average frame size of the system, and the average number of packets successfully delivered to the receiver in a frame is given by:

$$\begin{aligned} E\{\text{frame size}\} &= 2 \cdot P_{\text{zigzag}} + 1 \cdot (1 - P_{\text{zigzag}}) \\ &= 1 + P_{\text{zigzag}} \end{aligned} \tag{3}$$

$$E\{\text{\# success packets in a frame}\} = 1 \cdot P_1 + 2 \cdot P_{\text{zigzag}} \tag{4}$$

For each frame $k \in \{1, 2, ...\}$, let R_k and T_k respectively represent the number of packet successes in frame k and the size of frame k. Thus $\{R_k\}$ and $\{T_k\}$ are *i.i.d* sequences. Let t_n represent the ending time of the n^{th} frame, and let $N(t_n)$ represent the total number of successes up to time t_n. Define μ as the time average throughput. By renewal/reward theory, we have:

$$\begin{aligned} \mu \equiv \lim_{n\to\infty} \frac{N(t_n)}{t_n} &= \frac{\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} R_k}{\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} T_k} = \frac{E\{R\}}{E\{T\}} \\ &= \frac{E\{\text{\# success packets in a frame}\}}{E\{\text{frame size}\}} \quad \text{with prob. 1} \end{aligned} \tag{5}$$

Thus,

$$\mu = \frac{P_1 + 2P_{\text{zigzag}}}{1 + P_{\text{zigzag}}} = \frac{2Nq(1-q)^{N-2}(1 + (N-2)q)}{2 + N(N-1)q^2(1-q)^{N-2}} \tag{6}$$

To maximize throughput, we need to solve $\frac{\partial \mu}{\partial q} = 0$. But it is challenging to derive a closed-form formula for optimal q from this. Hence, we take a different approach to obtain the maximum achievable throughput.

It can be shown that the optimal transmission probability is $\theta(1/N)$, and hence we would like to find the optimal throughput by guessing the optimal transmission probability as

$$q^\star = \frac{\alpha}{N + \delta} \tag{7}$$

and optimizing α and δ for the maximum real throughput μ as N goes to infinity. Intuitively, the significant order of the optimal transmission probability must be N^{-1} since any other order will result in throughput of zero as N goes to infinity; higher order will result in infinite repeated collision, and lower order will result in too little transmission relative to N.

Substituting $q^\star$ into throughput μ in Eq.(6) and as $N \to \infty$,

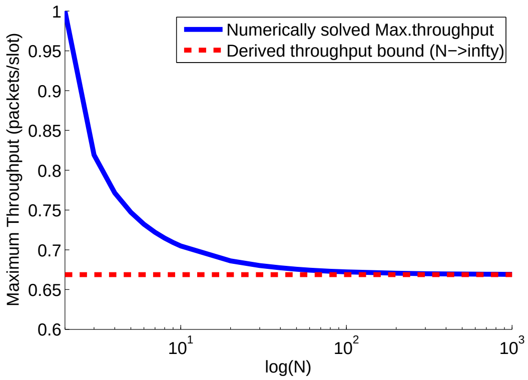

Fig. 2: Maximum throughput at various N, and the throughput bound when $N \to \infty$

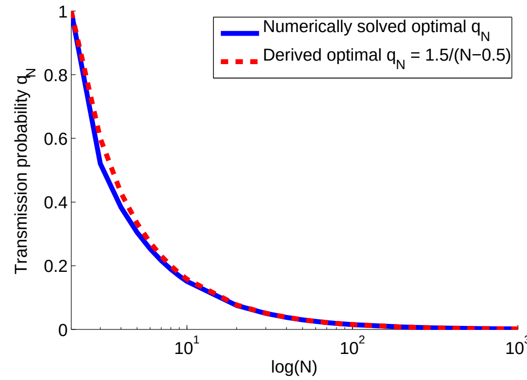

Fig. 3: Transmission probability $q^\star$ that achieves maximum throughput at various N

the throughput becomes,

$$\mu_N^\star \equiv \frac{2N\frac{\alpha}{N+\delta}\left(1-\frac{\alpha}{N+\delta}\right)^{N-2}\left(1+(N-2)\frac{\alpha}{N+\delta}\right)}{2+N(N-1)\left(\frac{\alpha}{N+\delta}\right)^2\left(1-\frac{\alpha}{N+\delta}\right)^{N-2}} \tag{8}$$

$$\lim_{N\to\infty}\mu_N^\star = \frac{2\alpha(1+\alpha)e^{-\alpha}}{2+\alpha^2e^{-\alpha}} \tag{9}$$

Optimizing α for maximum throughput $\mu^\star$ using numerical calculation results in $\alpha = 1.4995 \approx 1.5$. Also, in our model with ZIGZAG decoding, the transmission probability q should be 1 for $N \leq 2$ to maximize throughput since two simultaneous transmissions can always result in average throughput of 1 packet/slot. Hence we set $\delta = -0.5$. Thus, as $N \to \infty$, the optimal transmission probability is,

$$\boxed{q^\star = \frac{1.5}{N-0.5}} \tag{10}$$

and the bound on the maximum throughput is,

$$\boxed{\mu^\star \equiv \lim_{N\to\infty}\mu_N^\star \approx 0.6688} \tag{11}$$

Now, we verify the result of the above derivation by numerically solving the following two equations together at various N values to find the optimal throughput. Using the value of μ from Eq.(6), we solve:

$$\frac{\partial\mu}{\partial q} = 0, \quad \mu = \lambda$$

Fig.2 depicts the numerically calculated maximum throughput with ZIGZAG decoding at various N values, along with the derived throughput bound when $N \to \infty$. The figure shows that the numerically calculated maximum throughput does converge to the derived throughput bound as N goes to infinity. Also, Fig.3 plots the corresponding numerically calculated optimal packet transmission probability at various N values. This is closely overlapped with the packet transmission probability $q^\star$ derived in (Eq.(10)). These two results together confirm that our derivation is accurate.

Thus, in this N-user slotted random access system with ZIGZAG decoding, the maximum achievable throughput is 0.6688. Comparing this result to the throughput of this simple random access system without ZIGZAG decoding ($e^{-1} = 0.3678$, presented in [2]) shows that ZIGZAG can improve the throughput of the system by 81.8%.

III. Stabilized Slotted Aloha

In this section, we look at a stabilized version of Slotted Aloha [11] with ZIGZAG decoding. In this model, there are an infinite number of users each with at most one packet to send. New users arrive according to a Poisson process with rate λ packets/slot, and they transmit immediately on the slot in which they arrive (even if it is the second slot of a ZIGZAG frame). If the transmission results in a collision, then each user sending one of the colliding packets discovers the collision at the end of the slot and becomes a backlogged user. Backlogged users retransmit their packet independently every slot with some adaptive probability q_n, $(0 \leq q_n \leq 1)$, where n is the number of backlogged users in the system. If the transmitted packet is received at the receiver, then that backlogged user is removed from the system. Otherwise, it remains backlogged.

Let $N(k)$ be the number of backlogged users at the beginning of the k^{th} frame, and suppose that all backlogged users know the value of $N(k)$. At the beginning of the k^{th} frame, if $N(k) = n$, then all backlogged users transmit with probability q_n (and new users arrive independently according to a Poisson distribution of rate λ). Feedback is given in the same way as before at the end of the first slot of the frame (see Sec.II). The frame has size 1 if the feedback is '0,' '1,' or 'C.' In the case of a 'ZIGZAG' feedback, the frame is size 2, and the same two users who collided re-transmit the same packets in the second slot of the frame. All other backlogged users remain silent.

However, there is a chance that a new arrival on the second slot of the frame also transmits, in which case we get a feedback of 'C,' and we erase all information we had about the ZIGZAG event. That is, a ZIGZAG frame delivers exactly 2 packets in 2 slots if there are no new arrivals at the second

Fig. 4: A timeline of events in slotted Aloha. Two ZIGZAG frames are ruined by new arrivals on the second slot.

slot of the ZIGZAG frame, but delivers exactly 0 packets in two slots if there is a new arrival at the second slot of the ZIGZAG frame (see Fig. 4). Recall that new arrivals that are not successfully delivered are added to the set of backlogged users at the end of the frame.

We first define the probabilities of '1' (success) and 'ZIGZAG', given that $N(k) = n$, as follows:

$$P_1(n) = \lambda e^{-\lambda}(1-q_n)^n + e^{-\lambda} n q_n (1-q_n)^{n-1} \tag{12}$$

$$P_{\text{zigzag}}(n) = \frac{\lambda^2}{2} e^{-\lambda}(1-q_n)^n + \lambda e^{-\lambda}\binom{n}{1} q_n (1-q_n)^{n-1} + e^{-\lambda}\binom{n}{2} q_n^2 (1-q_n)^{n-1} \tag{13}$$

P_1 is the probability of the '1' event where exactly 1 packet attempted transmission at the beginning of a frame. P_{zigzag} is the probability of the 'ZIGZAG' event where exactly 2 packets attempted transmission at the beginning of a frame, but this event *does not* always mean that 2 packets are successfully delivered to the receiver. In fact, since a new arrival may attempt transmission in the second slot of a ZIGZAG frame and corrupt the ZIGZAG delivery (Fig. 4), the probability of 2 packets being successfully delivered is,

$$\begin{aligned} P_2(n) &= Pr\left\{\text{No new arrival \& ZIGZAG event} \mid N(k)=n\right\} \\ &= e^{-\lambda} P_{\text{zigzag}}(n) \end{aligned} \tag{14}$$

Again, the average frame size in this system is given by,

$$\begin{aligned} E\{\text{frame size} \mid N(k)=n\} &= 2\cdot P_{\text{zigzag}}(n) \\ &\quad + 1\cdot(1-P_{\text{zigzag}}(n)) \\ &= 1 + P_{\text{zigzag}}(n) \end{aligned}$$

and the average number of packets successfully delivered to the receiver in a frame is given by,

$$\begin{aligned} &E\{\#\ \text{packets delivered in a frame} \mid N(k)=n\} \\ &\qquad = 1\cdot P_1(n) + 2\cdot P_2(n) \\ &\qquad = P_1(n) + 2e^{-\lambda}P_{\text{zigzag}}(n) \end{aligned}$$

Now, consider the Discrete Time Markov Chain $N(k)$, and define the drift D_n as:

$$\begin{aligned} D_n &\equiv E\left\{N(k+1)-N(k) \mid N(k)=n\right\} \\ &= E\{\text{arrivals} \mid N(k)=n\} - E\{\text{departures} \mid N(k)=n\} \\ &= \lambda(1+P_{\text{zigzag}}(n)) - (P_1(n)+2P_2(n)) \\ &= \lambda(1+P_{\text{zigzag}}(n)) - (P_1(n)+2e^{-\lambda}P_{\text{zigzag}}(n)) \end{aligned} \tag{15}$$

By the Drift Theorem [12], [13], the system is stable if there is an $\epsilon > 0$ such that the drift D_n satisfies $D_n \le -\epsilon$ for large n (and unstable if $D_n > 0$ for large n). By Eq.(15), $D_n < 0$ is equivalent to the following expression:

$$\lambda < \frac{P_1(n) + 2e^{-\lambda}P_{\text{zigzag}}(n)}{1+P_{\text{zigzag}}(n)} \tag{16}$$

We now design the transmission probability $q_n^\star$ so that the right hand side of Eq.(16) converges to a constant $\mu^\star$ as $n \to \infty$. Thus, from Eq.(16), the system is stable (with negative drift for large n) whenever $\lambda < \mu^\star$. To this end, define $\mu_n^\star$ as the right hand side of (16), and define $q_n^\star$ as follows:

$$q_n^\star = \frac{\alpha - \lambda}{n - \delta} \tag{17}$$

such that nq_n is bounded for large n. The constants α and δ in Eq.(17) will be optimally chosen. By substituting $q_n^\star$ into the right hand side of Eq.(16), and by approximating $(1-q_n)^n \approx (1-q_n)^{n-1} \approx (1-q_n)^{n-2}$ and sending $n \to \infty$, we get,

$$\mu^\star \equiv \lim_{n\to\infty} \mu_n^\star = \frac{\alpha e^{-\alpha} + \alpha^2 e^{-\alpha} e^{-\lambda}}{1 + \frac{\alpha^2}{2} e^{-\alpha}} \tag{18}$$

To find the maximum throughput bound, we numerically solve the following two equations to optimize α:

$$\frac{\partial \mu^\star}{\partial \alpha} = 0, \quad \mu^\star = \lambda \tag{19}$$

to obtain the optimal $\alpha^\star = 1.310$. Hence, the optimal transmission probability $q_n^\star$ is,

$$q_n^\star = \frac{1.31 - \lambda}{n - \lambda - 0.69} \tag{20}$$

where δ has been set so that $q_n^\star = 1$ when $n = 2$. This gives us the optimal maximum throughput bound of

$$\boxed{\mu^\star = \lim_{n\to\infty} \mu_n^\star = 0.5123} \tag{21}$$

Thus in a stabilized slotted Aloha system, ZIGZAG decoding improves maximum throughput from $e^{-1} = 0.3678$ to 0.5123. This is a 39% improvement compared to the system without ZIGZAG decoding. However, this improvement result is not as much as that of the N-user random access model in Sec.II. The reason is because the system above suffers from a loss of throughput if a new user arrives to the system on the second slot of a ZIGZAG frame. Thus, we can improve the system throughput by modifying the above random access protocol such that the new arrivals during the first slot of a ZIGZAG frame do not transmit immediately (in the second slot of the

Fig. 5: A timeline showing the state transitions in CSMA Aloha with ZIGZAG

ZIGZAG frame in which they arrived). They listen to hear feedback at the end of the slot in which they arrive, and are then added to the set of 'backlogged' packets without transmitting on the next slot if they hear a 'ZIGZAG' on the slot in which it arrived. Thus, a ZIGZAG event will always end with 2 successful packet receptions.

Then the above analysis should be slightly modified such that,

$$(14) \longrightarrow \hat{P}_2(n) = P_{\text{zigzag}}(n) \tag{22}$$

$$(15) \longrightarrow \hat{D}_n = \lambda(1 + P_{\text{zigzag}}(n)) - (P_1(n) + 2P_{\text{zigzag}}(n)) \tag{23}$$

$$(16) \longrightarrow \hat{\mu}_n = \frac{P_1(n) + 2P_{\text{zigzag}}(n)}{1 + P_{\text{zigzag}}(n)} \tag{24}$$

$$(18) \longrightarrow \hat{\mu}^\star = \frac{\alpha e^{-\alpha} + \alpha^2 e^{-\alpha}}{1 + \frac{\alpha^2}{2} e^{-\alpha}} \tag{25}$$

and numerically solving Eq.(19) with these modified equations results in optimal $\hat{\alpha}^\star = 1.3558$. Thus, the optimal transmission probability $\hat{q}^\star$ is given as follows:

$$\hat{q}_n^\star = \frac{1.3558 - \lambda}{n - \lambda - 0.6442} \tag{26}$$

where δ has been set so that $\hat{q}^\star = 1$ when $n = 2$. This gives the optimal maximum throughput bound of:

$$\boxed{\hat{\mu}^\star = \lim_{n\to\infty} \hat{\mu}_n^\star = 0.6688} \tag{27}$$

The above result is identical to the result from the N-user random access system in Sec.II; ZIGZAG *decoding improves maximum throughput from* 0.3678 *to* 0.6688*, an* 81.8% *improvement compared to the system without* ZIGZAG *decoding in stabilized slotted Aloha*.

Although our analyses do not consider conflict resolution techniques such as [6], [7] which provide maximum thoughput of 0.43 and 0.4871 respectively, we believe that ZIGZAG decoding will improve throughput bounds in those cases as well. Since the essence of ZIGZAG decoding is to allow correct reception of 2 colliding packets, it is complementary to the conflict resolution techniques and will only increase the maximum throughput in addition to the effect of the conflict resolution themselves.

Finally, we have also verified this maximum achievable throughput using packet-level simulations. In the simulation, we have used $\hat{q}_n^\star$ in Eq.(26) as the adaptive transmission probability (recall that n is the number of backlogged packets in the system), and each simulation ran for 100,000 packets with maximum backlog queue size of 500. Specifically, each simulation was run for a fixed value of λ, and the value of λ was increased for each new simulation to find the maximum λ for which there are no queue overflow events. The simulation resulted in maximum throughput of 0.6675. This result is almost identical to the optimal throughput obtained above, and confirms that our analysis is accurate.

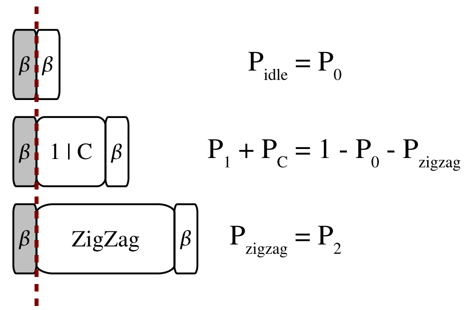

Fig. 6: Three possible state transition times in CSMA Aloha with ZIGZAG: β to 'β,' '1 or C,' 'ZIGZAG'

IV. CSMA SLOTTED ALOHA

In this section, we examine the maximum throughput of slotted non-persistent CSMA [5] (Carrier Sense Multiple Access) system. This system is almost identical to the slotted Aloha system in the previous section (Section III) with the following differences. The major difference between CSMA slotted Aloha and ordinary slotted Aloha is that idle slots in CSMA are 'Mini-slots' with a duration β units of time ($\beta \leq 1$) (See Fig. 5). Newly arriving users during an idle slot (mini-slot) will attempt transmission in the next slot. If a new user with a packet arrives while transmissions are in progress, (either success, ZIGZAG, or collision) they are added to the set of backlogged users. Backlogged users wait politely until they hear a full mini-slot, then attempt transmission with probability q_n.

To analyze the CSMA Aloha, we can use a Markov chain again, using the number n of backlogged packets as the state and the ends of idle slots as the state transition times. Note that each busy (success, ZIGZAG, or collision) frame must be followed by an idle slot, since nodes are allowed to start transmission only after detecting an idle slot.

In this system, there are four different types of states; idle, success-idle, collision-idle, and ZIGZAG-idle. The time between successive state transitions is either β (in the case of an idle slot), or $1 + \beta$ (in the case of success or collision slot followed by an idle), or $2 + \beta$ (in the case of ZIGZAG slots followed by an idle) (see Fig.6).

First, we define the probabilities of each of four events:

$$P_0(n) = e^{-\lambda\beta}(1-q_n)^n \quad (28)$$
$$P_1(n) = \lambda\beta e^{-\lambda\beta}(1-q_n)^n + e^{-\lambda\beta} n q_n (1-q_n)^{n-1} \quad (29)$$
$$P_{\text{zigzag}}(n) = \frac{(\lambda\beta)^2}{2} e^{-\lambda\beta}(1-q_n)^n + \lambda\beta e^{-\lambda\beta}\binom{n}{1} q_n (1-q_n)^{n-1} + e^{-\lambda\beta}\binom{n}{2} q_n^2 (1-q_n)^{n-2} \quad (30)$$
$$P_C(n) = 1 - P_0(n) - P_1(n) - P_{\text{zigzag}}(n) \quad (31)$$

The expected time between successive state transitions and the expected number of arrivals and departures are given as:

$$\begin{aligned} E\{\text{transition time}\} &= \beta + 1\cdot(P_1(n)+P_C(n)) + 2\cdot P_{\text{zigzag}}(n) \\ &= \beta + 1\cdot(1-P_0(n)-P_{\text{zigzag}}(n)) \\ &\quad + 2\cdot P_{\text{zigzag}}(n) \\ &= \beta + 1 - P_0(n) + P_{\text{zigzag}}(n) \quad (32) \\ E\{\text{arrivals}\} &= \lambda \cdot E\{\text{transition time}\} \\ &= \lambda(\beta + 1 - P_0(n) + P_{\text{zigzag}}(n)) \quad (33) \\ E\{\text{departures}\} &= P_1(n) + 2\cdot P_{\text{zigzag}}(n) \quad (34) \end{aligned}$$

Then we can obtain the drift D_n using Eq.(33)(34) as follows:

$$\begin{aligned} D_n &\equiv E\{N(k+1) - N(k) | N(k) = n\} \\ &= E\{\text{arrivals}\} - E\{\text{departures}\} \\ &= \lambda(\beta + 1 - P_0(n) + P_{\text{zigzag}}(n)) - (P_1(n) + 2P_{\text{zigzag}}(n)) \end{aligned} \quad (35)$$

From the above equation, we can see that the drift in state n is negative if the following holds:

$$\lambda < \frac{P_1(n) + 2P_{\text{zigzag}}(n)}{\beta + 1 - P_0(n) + P_{\text{zigzag}}(n)} \quad (36)$$

To keep the system stable, the drift D_n must satisfy $D_n < 0$ for large n. Hence, the right hand side of Eq.(36) is the bound on maximum throughput:

$$\mu \equiv \text{Max.Throughput} \equiv \lim_{n\to\infty}\left[\frac{P_1(n) + 2P_{\text{zigzag}}(n)}{\beta + 1 - P_0(n) + P_{\text{zigzag}}(n)}\right]$$

Suppose the packet transmission probability has the following structure:

$$q_n \equiv \frac{\alpha - \lambda\beta}{n} \quad (37)$$

where α is a constant to be determined. In the above equation, $\alpha \equiv \lambda\beta + nq_n$ is the expected number of attempted transmissions following a transition to state n. For large n, we can approximate the state transition probabilities in Eq.(28)(29)(30) as,

$$\begin{aligned} P_0 &\approx e^{-\alpha} \\ P_1 &\approx \alpha e^{-\alpha} \\ P_{\text{zigzag}} &\approx \frac{\alpha^2}{2} e^{-\alpha} \end{aligned}$$

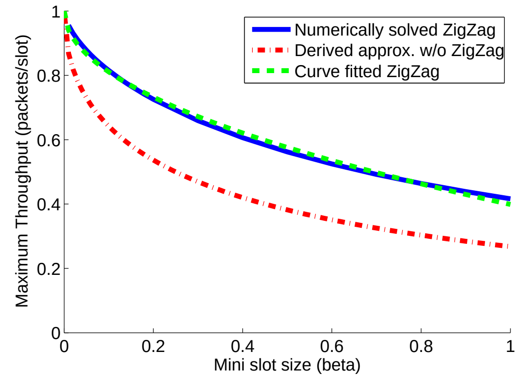

Fig. 7: Maximum throughput bound of the CSMA system with and without ZIGZAG at different β values

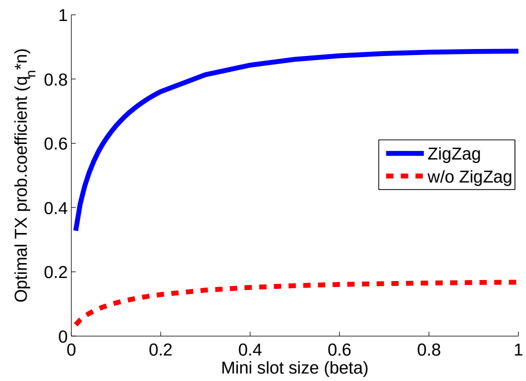

Fig. 8: Optimal aggregate transmission probability $q_n * n$ (with and without ZIGZAG) at different β values for the CSMA system

and thus we can approximate this maximum throughput μ as follows:

$$\mu \approx \frac{\alpha e^{-\alpha} + \alpha^2 e^{-\alpha}}{\beta + 1 - e^{-\alpha} + \frac{\alpha^2}{2} e^{-\alpha}} \quad (38)$$

by approximating $(1-q_n)^n \approx (1-q_n)^{n-1} \approx (1-q_n)^{n-2}$ and sending $n \to \infty$.

To find the bound on maximum throughput μ, we optimize α for large n by solving,

$$\frac{\partial\mu}{\partial\alpha} = 0 \quad (39)$$
$$\mu = \lambda \quad (40)$$

Unfortunately, it is difficult to solve Eq.(39) to obtain an intuitive closed form formula. Hence, we numerically solve Eq.(39) and Eq.(40) together at various β values to obtain the maximum throughput bound.

Fig.7 depicts the numerically calculated maximum throughput with ZIGZAG along with the approximated throughput without ZIGZAG ($\beta + 1 - \sqrt{\beta^2 + 2\beta}$) at various β values.[2] Also, Fig.8 depicts the numerically calculated optimal packet

[2]Analysis in [2] presents $1 - \sqrt{2\beta}$ as the approximate maximum throughput, but that came from approximating $\beta + 1 - \sqrt{\beta^2 + 2\beta}$

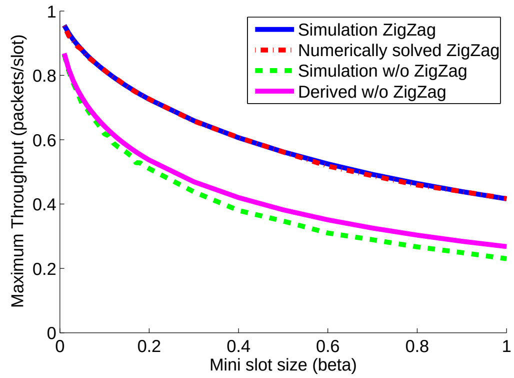

Fig. 9: Maximum throughput results (with and without ZIGZAG) from simulations at different β values for the CSMA system

transmission probability at various β values. Curve fitting the numerically calculated maximum throughput values gives us the following expression for the maximum throughput with ZIGZAG:

$$\boxed{\mu = 1 - 0.5966\sqrt{\beta} - 0.0045\beta} \quad (41)$$

We have verified this maximum achievable throughput at various β values using packet-level simulations. In the simulation, we have used the numerically obtained optimal transmission probabilities (in Fig.8), and each simulation ran for 100,000 packets with maximum backlog queue size of 500.

Fig.9 shows the maximum throughput achieved in the simulation with and without ZIGZAG decoding. The figure also includes the numerically calculated throughput bounds in Fig.7 for convenience of comparison. The simulation results clearly show that the maximum throughput achieved by using ZIGZAG decoding is greater than the case where ZIGZAG decoding is not used. For example when $\beta = 0.01$, the maximum throughput with ZIGZAG is 0.9645 and without ZIGZAG is 0.8682 which corresponds to 11.1% improvement. When $\beta = 0.1$, they are 0.8122, 0.6417, and 26.5% respectively. Furthermore, the figure also shows that the simulation results for both with and without ZIGZAG are almost identical to the numerically calculated maximum throughput for the case with ZIGZAG decoding and derived approximate maximum throughput without ZIGZAG ($\beta+1-\sqrt{\beta^2+2\beta}$), respectively. For the case without ZIGZAG decoding, the simulation result is very close to the calculated approximated throughput when β is small, but diverges a little when when β grows. This is because the derivation of $\mu \approx \beta + 1 - \sqrt{\beta^2 + 2\beta}$ used an assumption that $e^{-x} \approx 1 - x + \frac{x^2}{2}$ for $\beta \ll 1$ while performing the approximation.

V. CONCLUSION

We have presented throughput bounds for multi-user random access systems with ZIGZAG decoding [1]. Our analysis and simulation results have shown that ZIGZAG decoding can significantly improve the maximum throughput of the random access system. Table I summarizes the main results. Although

Model	w/o ZIGZAG	with ZIGZAG	% gain
Random Access	0.3678	0.6688	81.8
Aloha	0.3678	0.6688	81.8
CSMA ($\beta = 0.1$)	0.6417	0.8122	26.5
CSMA ($\beta = 0.05$)	0.7298	0.8759	20.0

TABLE I: Maximum throughput bounds with and without ZIGZAG decoding

the analysis has been done in simplified models, we believe that the same trend will hold in real systems also. We believe that our results with ZIGZAG decoding motivates a MAC design with more aggressive transmission probabilities that exploits concurrent transmissions to maximize throughput.

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