Pseudo-Weights

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Overview

- ML Decoding Decision Boundaries
- LP Decoding Decision Boundaries
- Pseudo-Weights
- Pseudo-Weight Spectra

ML Decoding vs. LP Decoding

- Maximum-likelihood (ML) decoding:

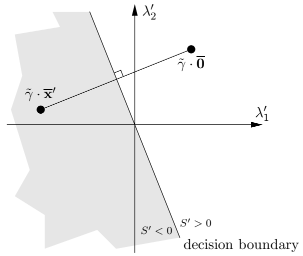
$$\hat{\mathbf{x}} = \arg \min_{\mathbf{x} \in \mathrm{conv}\left(\cap_{j=1}^{m} \mathcal{C}_j\right)} \sum_{i=1}^{n} \gamma_i x_i.$$

- Linear programming (LP) decoding:

$$\hat{\boldsymbol{\omega}} = \arg \min_{\boldsymbol{\omega} \in \cap_{j=1}^{m} \mathrm{conv}(\mathcal{C}_j)} \sum_{i=1}^{n} \gamma_i \omega_i.$$

ML Decoding Decision Boundaries

Decision regions under MLD when only the zero codeword is competing against the codeword $\mathbf{x}'$. (For any codeword $\mathbf{x}'$, we defined $\overline{\mathbf{x}}' \triangleq 1 - 2\mathbf{x}'$.)



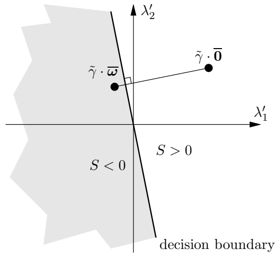
$$S' \triangleq \langle \mathbf{x}', \boldsymbol{\Lambda} \rangle - \langle \mathbf{0}, \boldsymbol{\Lambda} \rangle = \sum_{i \in \mathcal{I}:\, x'_i = 1} \Lambda_i$$

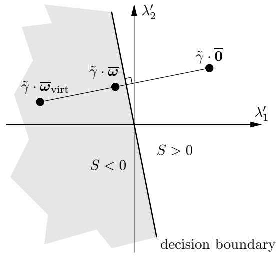
$$S' \big|_{\mathbf{x}=\mathbf{0}\ \text{sent}} \sim \mathcal{N}\left(4R\frac{E_\mathrm{b}}{N_0} w_\mathrm{H}(\mathbf{x}'),\ 8R\frac{E_\mathrm{b}}{N_0} w_\mathrm{H}(\mathbf{x}') \right)$$

Note: the MLD decision boundary only goes through the origin if the modulated versions of the two competing codewords have the same energy.

LP Decoding Decision Boundaries

Decision regions under LPD when only the zero codeword is competing against the pseudo-codeword $\boldsymbol{\omega}$. (We defined $\overline{\boldsymbol{\omega}} \triangleq 1 - 2\boldsymbol{\omega}$.)





$$\boldsymbol{\omega}_{\text{virt}} \triangleq \frac{\|\boldsymbol{\omega}\|_1}{\|\boldsymbol{\omega}\|_2^2} \cdot \boldsymbol{\omega}$$

$\boldsymbol{\omega}_{\text{virt}}$ is defined such that the decision hyperplane is at the same Euclidean distance from $\tilde{\gamma} \cdot \overline{\mathbf{0}}$ and from $\tilde{\gamma} \cdot \overline{\boldsymbol{\omega}}_{\text{virt}}$.

LP Decoding Decision Boundaries

$$S \triangleq \langle \boldsymbol{\omega}, \boldsymbol{\Lambda} \rangle - \langle \mathbf{0}, \boldsymbol{\Lambda} \rangle = \sum_{i \in \mathcal{I}} \omega_i \Lambda_i$$

$$S|_{\mathbf{X}=\mathbf{0}\ \text{sent}} \sim \mathcal{N}\left(4R\frac{E_{\text{b}}}{N_0} \sum_{i \in \mathcal{I}} \omega_i,\ 8R\frac{E_{\text{b}}}{N_0} \sum_{i \in \mathcal{I}} \omega_i^2 \right)$$

WGNC Pseudo-Weight

$$w_{\text{p}}^{\text{AWGNC}}(\boldsymbol{\omega}) \triangleq \frac{\|\boldsymbol{\omega}\|_1^2}{\|\boldsymbol{\omega}\|_2^2} = \frac{\left(\sum_{i \in [n]} \omega_i\right)^2}{\sum_{i \in [n]} \omega_i^2}$$

BSC Pseudo-Weight

Let $\boldsymbol{\omega} \in \mathbb{R}_+^n$. Let $\boldsymbol{\omega}'$ be a vector of length n with the same components as $\boldsymbol{\omega}$ but in non-increasing order. Introducing

$$f(\xi) \triangleq \omega'_i \quad (i - 1 < \xi \le i,\ 0 < \xi \le n),$$
$$F(\xi) \triangleq \int_0^{\xi} f(\xi')\mathrm{d}\xi',$$
$$e \triangleq F^{-1}\left(\frac{F(n)}{2}\right),$$

the BSC pseudo-weight $w_{\text{p}}^{\text{BSC}}(\boldsymbol{\omega})$ is defined to be $w_{\text{p}}^{\text{BSC}}(\boldsymbol{\omega}) \triangleq 2e$.
If $\boldsymbol{\omega} = \mathbf{0}$ then $w_{\text{p}}^{\text{BSC}}(\boldsymbol{\omega}) = 0$.

Note that the quantity e is obviously related to the median of the "pdf" given by $f(\xi)/F(n)$.

BEC Pseudo-Weight

$$w_{\mathrm{p}}^{\mathrm{BEC}}(\boldsymbol{\omega}) = |\mathsf{supp}(\boldsymbol{\omega})|.$$

Fractional and Max-Fractional Weight

$$w_{\mathrm{frac}}(\boldsymbol{\omega}) = \|\boldsymbol{\omega}\|_1 .$$

$$w_{\mathrm{max-frac}}(\boldsymbol{\omega}) \triangleq \frac{w_{\mathrm{frac}}(\boldsymbol{\omega})}{\|\boldsymbol{\omega}\|_\infty} = \frac{\|\boldsymbol{\omega}\|_1}{\|\boldsymbol{\omega}\|_\infty}.$$

For $\boldsymbol{\omega} = \mathbf{0}$ we define $w_{\mathrm{max-frac}}(\boldsymbol{\omega}) \triangleq 0$.

Note that we use a slightly different notation than [Feldman:03]. Here, w_{frac} and $w_{\mathrm{max-frac}}$ are defined for any vector in $\mathbb{R}_+^n$, whereas in [Feldman:03], w_{frac} and $w_{\mathrm{max-frac}}$ already denote the minimum of these values over all nonzero vertices of the fundamental polytope.

Minium Pseudo-Weights (Part 1)

The minimum AWGNC, BSC, and BEC pseudo-weight and the minimum fractional and max-fractional weights are defined to be, respectively,

$$\begin{aligned}
w_{\mathrm{p}}^{\mathrm{AWGNC,min}}(\mathbf{H}) &\triangleq \min_{\boldsymbol{\omega}\in\mathcal{V}(\mathcal{P}(\mathbf{H}))\setminus\{\mathbf{0}\}} w_{\mathrm{p}}^{\mathrm{AWGNC}}(\boldsymbol{\omega}),\\
w_{\mathrm{p}}^{\mathrm{BSC,min}}(\mathbf{H}) &\triangleq \min_{\boldsymbol{\omega}\in\mathcal{V}(\mathcal{P}(\mathbf{H}))\setminus\{\mathbf{0}\}} w_{\mathrm{p}}^{\mathrm{BSC}}(\boldsymbol{\omega}),\\
w_{\mathrm{p}}^{\mathrm{BEC,min}}(\mathbf{H}) &\triangleq \min_{\boldsymbol{\omega}\in\mathcal{V}(\mathcal{P}(\mathbf{H}))\setminus\{\mathbf{0}\}} w_{\mathrm{p}}^{\mathrm{BEC}}(\boldsymbol{\omega}),\\
w_{\mathrm{frac}}^{\mathrm{min}}(\mathbf{H}) &\triangleq \min_{\boldsymbol{\omega}\in\mathcal{V}(\mathcal{P}(\mathbf{H}))\setminus\{\mathbf{0}\}} w_{\mathrm{frac}}(\boldsymbol{\omega}),\\
w_{\mathrm{max-frac}}^{\mathrm{min}}(\mathbf{H}) &\triangleq \min_{\boldsymbol{\omega}\in\mathcal{V}(\mathcal{P}(\mathbf{H}))\setminus\{\mathbf{0}\}} w_{\mathrm{max-frac}}(\boldsymbol{\omega}),
\end{aligned}$$

where $\mathcal{V}(\mathcal{P}(\mathbf{H})) \setminus \{0\}$ is the set of all non-zero vertices of the fundamental polytope $\mathcal{P}(\mathbf{H})$.

Minium Pseudo-Weights (Part 2)

It is important to note that the above minimal pseudo-weights depend on the choice of parity-check matrix $\mathbf{H}$, i.e. different parity-check matrices for the same code can lead to different minimal weights.

This is in contrast to the minimum Hamming weight of a code which is independent of the specific choice of parity-check matrix by which a binary linear code is represented.

Minium Pseudo-Weights (Part 3)

Lemma

$$w_{\mathrm{p}}^{\mathrm{AWGNC,min}}(\mathbf{H}) = \min_{\boldsymbol{\omega}\in\mathcal{P}(\mathbf{H})\setminus\{\mathbf{0}\}} w_{\mathrm{p}}^{\mathrm{AWGNC}}(\boldsymbol{\omega}) = \min_{\boldsymbol{\omega}\in\mathcal{K}(\mathbf{H})\setminus\{\mathbf{0}\}} w_{\mathrm{p}}^{\mathrm{AWGNC}}(\boldsymbol{\omega}),$$

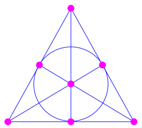
$$w_{\mathrm{p}}^{\mathrm{BSC,min}}(\mathbf{H}) = \min_{\boldsymbol{\omega}\in\mathcal{P}(\mathbf{H})\setminus\{\mathbf{0}\}} w_{\mathrm{p}}^{\mathrm{BSC}}(\boldsymbol{\omega}) = \min_{\boldsymbol{\omega}\in\mathcal{K}(\mathbf{H})\setminus\{\mathbf{0}\}} w_{\mathrm{p}}^{\mathrm{BSC}}(\boldsymbol{\omega}),$$

$$w_{\mathrm{p}}^{\mathrm{BEC,min}}(\mathbf{H}) = \min_{\boldsymbol{\omega}\in\mathcal{P}(\mathbf{H})\setminus\{\mathbf{0}\}} w_{\mathrm{p}}^{\mathrm{BEC}}(\boldsymbol{\omega}) = \min_{\boldsymbol{\omega}\in\mathcal{K}(\mathbf{H})\setminus\{\mathbf{0}\}} w_{\mathrm{p}}^{\mathrm{BEC}}(\boldsymbol{\omega}),$$

$$w_{\mathrm{max-frac}}^{\mathrm{min}}(\mathbf{H}) = \min_{\boldsymbol{\omega}\in\mathcal{P}(\mathbf{H})\setminus\{\mathbf{0}\}} w_{\mathrm{max-frac}}(\boldsymbol{\omega}) = \min_{\boldsymbol{\omega}\in\mathcal{K}(\mathbf{H})\setminus\{\mathbf{0}\}} w_{\mathrm{max-frac}}(\boldsymbol{\omega}).$$

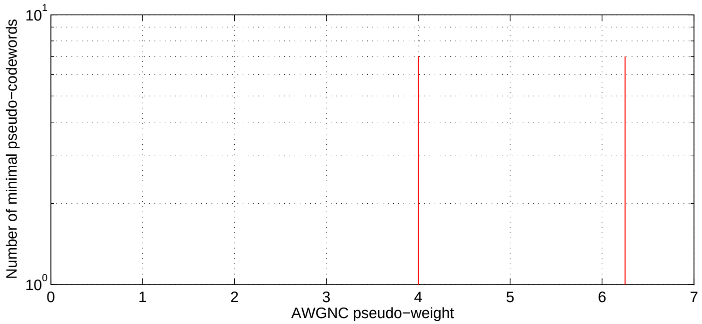
Note that there is no such statement for the fractional weight.

LP Decoding of Codes based on Finite Geometries



Consider the PG(2,2)-based $[7,3,4]$ binary linear code.

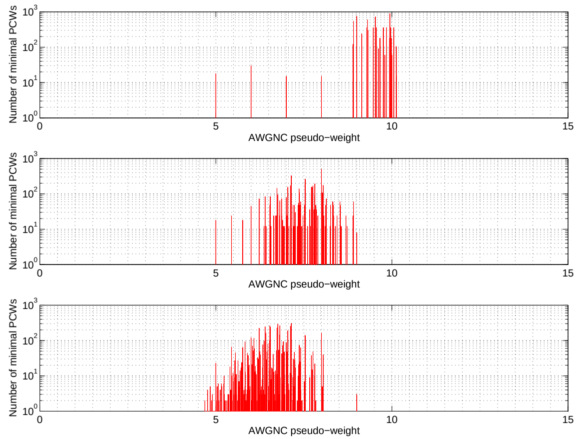
Here is its minimal pseudo-codeword spectrum:



LP Decoding of Codes based on Finite Geometries

Consider the EG(2,4)-based $[15,7,5]$ binary linear code.

Here are some minimal pseudo-codeword spectra for different parity-check matrices of this code:



PCM of size 15×15

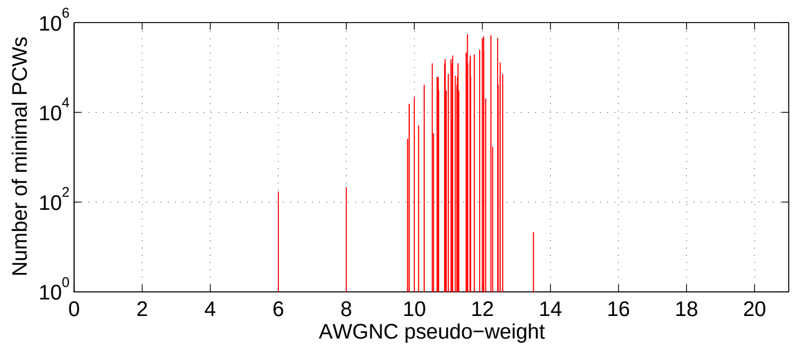
PCM of size 9×15

PCM of size 8×15

LP Decoding of Codes based on Finite Geometries

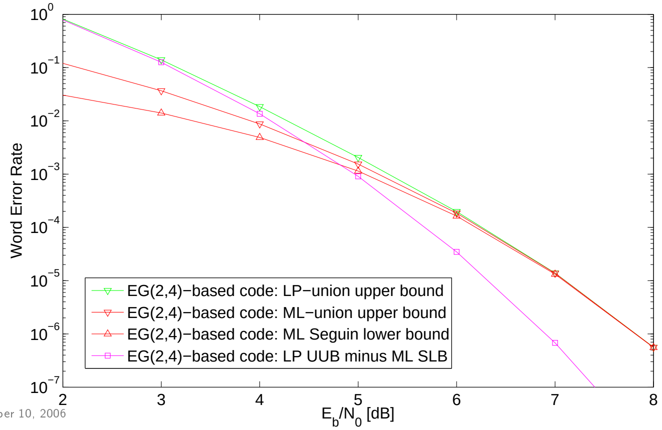
Consider the PG(2,4)-based $[21,11,6]$ binary linear code.

Here is its minimal pseudo-codeword spectrum:



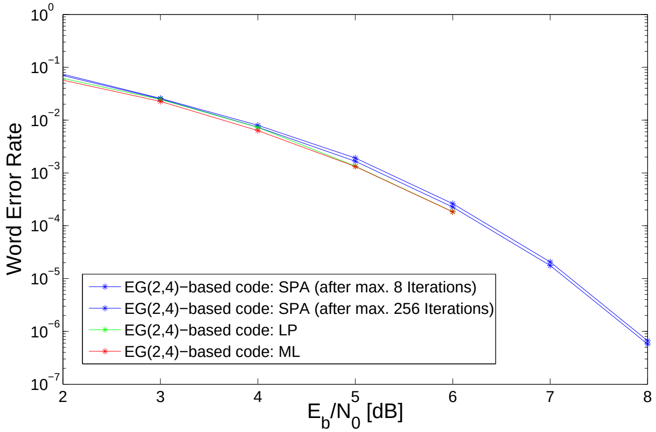
LP Decoding of Codes based on Finite Geometries

Consider the EG(2,4)-based $[15, 7, 5]$ binary linear code. The following plot shows upper and lower bounds on the word error rate of LP and ML decoding.



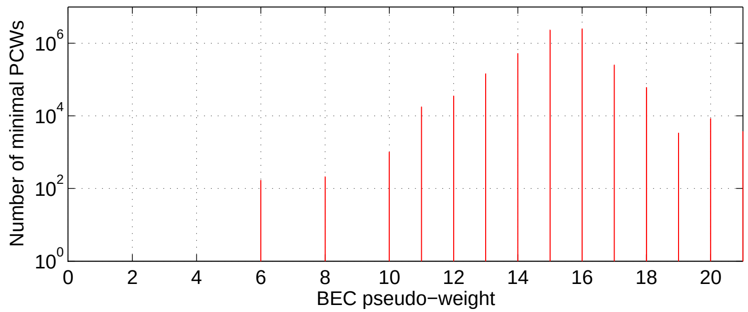
LP Decoding of Codes based on Finite Geometries

Consider the EG(2,4)-based $[15, 7, 5]$ binary linear code. The following plot shows the word error rate for different decoding algorithms.



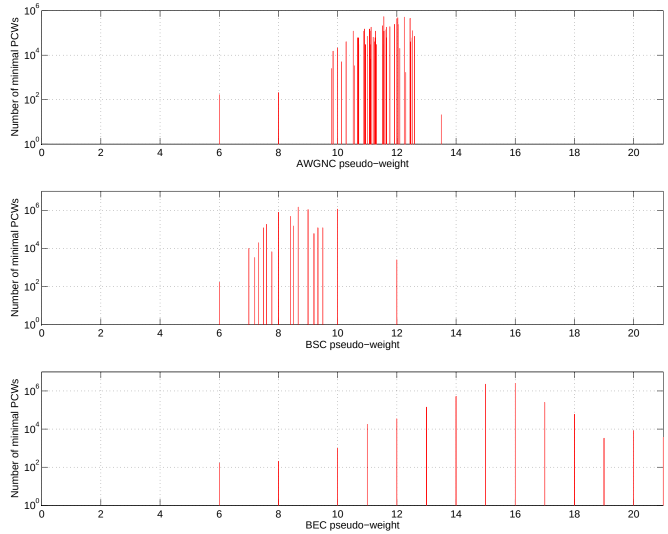
BEC Pseudo-Weight Spectrum

Consider the PG(2,4)-based $[21, 11, 6]$ binary linear code.
Here is its BEC pseudo-weight minimal pseudo-codeword spectrum:



Kashyap and Vardy (ISIT 2003) looked at a related issue: sizes of minimal stopping sets.

WGNC / BSC / BEC Pseudo-Weight Spectra



References

For more info, see e.g.

- P. O. Vontobel and R. Koetter, "Graph-cover decoding and finite-length analysis of message-passing iterative decoding of LDPC codes", submitted to IEEE Trans. on Inform. Theory, Dec. 2005. [http://www.arxiv.org/abs/cs.IT/0512078]

- R. Smarandache and P. O. Vontobel, "Pseudo-codeword analysis of Tanner graphs from projective and Euclidean planes", submitted to IEEE Trans. on Inform. Theory, Feb. 2006. [http://www.arxiv.org/abs/cs.IT/0602089]