On the Threshold of LP Decoding

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Overview

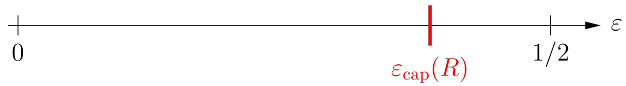
- Thresholds
- Bounds on the LP decoding threshold

Motivation (Part 1)

Assume that the channel is a BSC with cross-over probability ε.

Channel capacity:

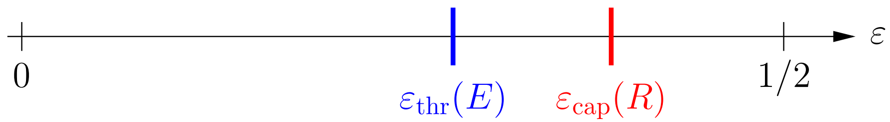
- Channel coding theorem
  (Gallager's random coding error exponent, etc.)
- Converse to the channel coding theorem
  (Fano's inequality, etc.)



Important: we are allowed to use the best available coding and decoding schemes for a given rate R.

Motivation (Part 2)

Assume that the channel is a BSC with cross-over probability ε.
Additionally, assume that we put restrictions on the coding schemes and/or on the decoding schemes.
$\Rightarrow$ Thresholds.



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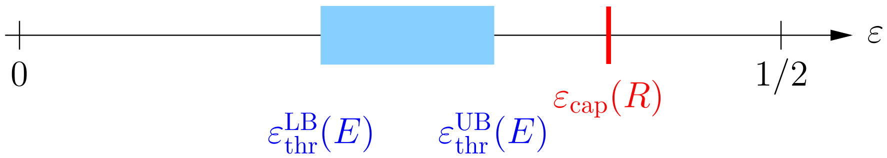


Existence of LP Decodin Thresholds

- A priori it is not clear for what families/ensembles of codes there is an LP decoding threshold.
- The tight connection between min-sum algorithm decoding and LP decoding suggest that families/ensembles that have a threshold under min-sum algorithm decoding also have a threshold under LP decoding.
- [Koetter:Vontobel:06]: there is an LP decoding threshold for $(w_{\text{col}}, w_{\text{row}})$-regular LDPC codes where $2 < w_{\text{col}} < w_{\text{row}}$.

BSC: n Upper Bound on the Threshold (Part 1)

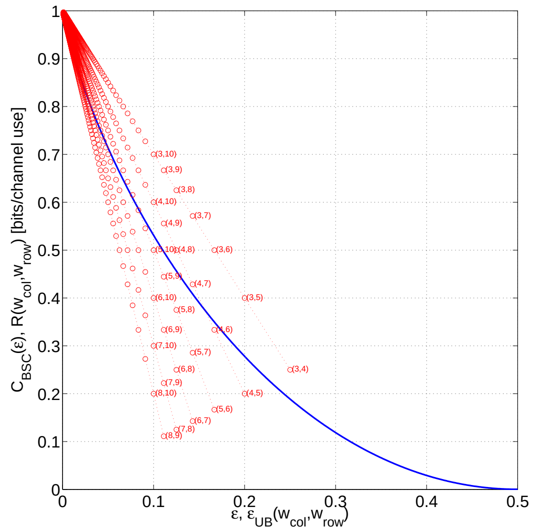
Theorem:

- Consider a family of $(w_{\text{col}}, w_{\text{row}})$-regular codes of increasing block length n.
- Consider a BSC with cross-over probability ε.
- In the limit $n \to \infty$, if

$$\varepsilon > \frac{1}{w_{\text{row}}}$$

then with probability 1 the LP decoder will not decode to the transmitted codeword.

BSC: n Upper Bound on the Threshold (Part 2)



BSC: n Upper Bound on the Threshold (Part 3)

Theorem: Consider a family of codes where the minimal row-degree goes to $w_{\mathrm{row}}^{\min}(\infty)$ when $n \to \infty$ and a BSC with cross-over probability ε. In the limit $n \to \infty$, if

$$\varepsilon > \frac{1}{w_{\mathrm{row}}^{\min}(\infty)}$$

then with probability 1 the LP decoder will not decode to the transmitted codeword.

Corollary: For any family of codes where $w_{\mathrm{row}}^{\min}(n)$ grows unboundedly, i.e. where

$$\lim_{n\to\infty} w_{\mathrm{row}}^{\min}(n) = \infty,$$

the above right-hand side expression goes to 0.

Not Decidin for the ll-Zeros Codeword (Part 1)

Linear programming (LP) decoding:

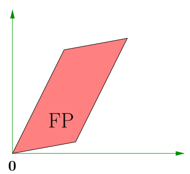
$$\hat{\boldsymbol{\omega}} = \arg \min_{\boldsymbol{\omega} \in \mathcal{P}(\mathbf{H})} \sum_{i=1}^{n} \gamma_i \omega_i.$$

Assume that the zero codeword has been sent. LP decoding does not decide for the all-zeros codeword if there is a vector

$$\boldsymbol{\omega} \in \mathcal{P}(\mathbf{H}) \setminus \{\mathbf{0}\}$$

such that

$$\sum_{i=1}^{n} \gamma_i \omega_i < 0.$$



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Assume that the zero codeword has been sent. LP decoding does not decides for the all-zeros codeword if there is a vector

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such that

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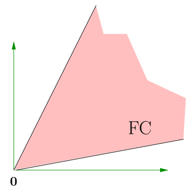

Not Decidin for the ll-Zeros Codeword (Part 3)

- Assume that we have a $(w_{\mathrm{col}}, w_{\mathrm{row}})$-regular LDPC code.
- Moreover, let $\boldsymbol{\omega} \in \mathbb{R}^n$ be a vector with the following entries:

$$\omega_i \triangleq \begin{cases} \frac{1}{w_{\mathrm{row}}-1} & \text{if } \gamma_i \geq 0 \\ 1 & \text{if } \gamma_i < 0 \end{cases}.$$

  One can easily verify that $\boldsymbol{\omega} \in \mathcal{K}(\mathbf{H})$.
- So, if

$$0 > \sum_{i=1}^{n} \gamma_i \omega_i = \left(\sum_{\substack{i=1 \\ \gamma_i \geq 0}}^{n} \gamma_i \right) \cdot \frac{1}{w_{\mathrm{row}} - 1} + \left(\sum_{\substack{i=1 \\ \gamma_i < 0}}^{n} \gamma_i \right) \cdot 1$$

  then LP decoding does not decide for the all-zeros codeword.

Not Decidin for the ll-Zeros Codeword: BSC (Part1)

- For simplicity, assume that we are transmitting over a BSC with crossover probability $0 \leq \varepsilon < 1/2$.

$$\Rightarrow \quad \gamma_i \in \{\pm G\} \quad \text{where} \quad G \triangleq \log\left(\frac{1-\varepsilon}{\varepsilon}\right) > 0.$$

- So, if

$$0 > \sum_{i=1}^{n} \gamma_i \omega_i = G \cdot \left((\#\text{not flipped}) \frac{1}{w_{\text{row}} - 1} - (\#\text{flipped}) \right).$$

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Not Decidin for the ll-Zeros Codeword: BSC (Part 2)

So, if

$$0 > \sum_{i=1}^{n} \gamma_i \omega_i = G \cdot \left((\#\text{not flipped}) \frac{1}{w_{\text{row}} - 1} - (\#\text{flipped}) \right).$$

then LP decoding does not decide for the all-zeros codeword.

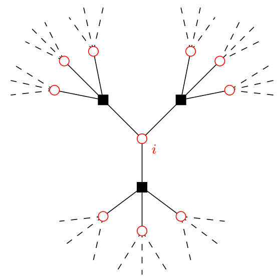
- Upon normalization, the above condition reads

$$0 > \frac{1}{n} \sum_{i=1}^{n} \gamma_i \omega_i = G \cdot \left(\frac{(\#\text{not flipped})}{n} \frac{1}{w_{\text{row}} - 1} - \frac{(\#\text{flipped})}{n} \right).$$

- In the limit $n \to \infty$, the above condition is with probability one equal to the condition

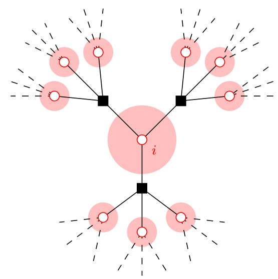
$$0 > \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \gamma_i \omega_i = G \cdot \left((1-\varepsilon) \frac{1}{w_{\text{row}} - 1} - \varepsilon \right).$$

0-Nei hborhood-Based Bounds (Part 1)



$\boldsymbol{\omega}$-vector that we constructed before: note that the the assignment of a value to ω_i was based only on the value of γ_i.

0-Nei hborhood-Based Bounds (Part 2)



$\boldsymbol{\omega}$-vector that we constructed before: note that the the assignment of a value to $\boldsymbol{\omega}_i$ was based only on the value of γ_i:

$$\omega_i = f(\gamma_i) = f\left(\{\gamma_{i'}\}_{i' \in \mathcal{N}_i^{(0)}} \right).$$

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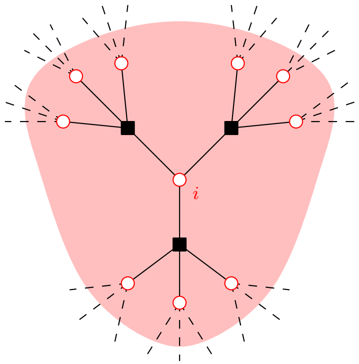
$$\omega_i = f(\gamma_i) = f\left(\{\gamma_{i'}\}_{i' \in \mathcal{N}_i^{(0)}}\right).$$

$$\omega_i \triangleq \begin{cases} \frac{1}{w_{\mathrm{row}}-1} & \text{if } \gamma_i \geq 0 \\ 1 & \text{if } \gamma_i < 0 \end{cases}.$$

One can easily check that $\boldsymbol{\omega} \in \mathcal{K}(\mathbf{H})$.

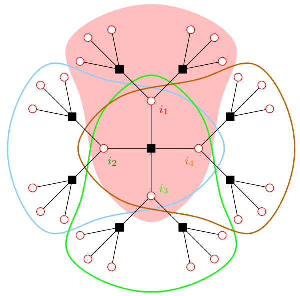
2-Nei hborhood-Based Bounds on the Threshold



Generalization:

$$\omega_i = f\left(\{\gamma_{i'}\}_{i' \in \mathcal{N}_i^{(2)}}\right).$$

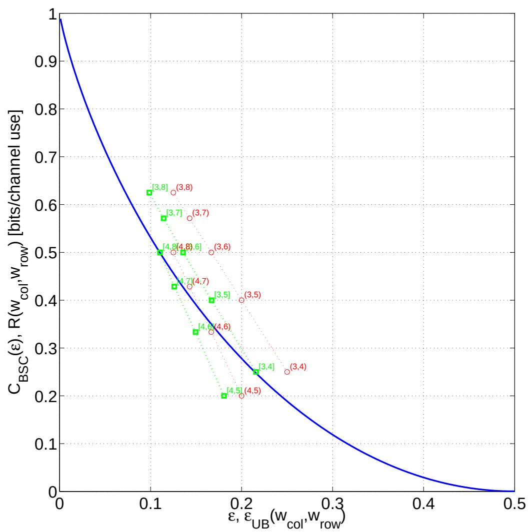
2-Nei hborhood-Based Bounds on the Threshold



We must take care of constrains: the map $f\left(\{\gamma_{i'}\}_{i' \in \mathcal{N}_i^{(2)}}\right)$ has to yield a vector in $\mathcal{K}(\mathbf{H})$.

$\Rightarrow$ We can set up a linear program that yields the best possible threshold for a 2-neighborhood. (Graph automorphisms help in simplifying that LP.)

2-Nei hborhood-Based Bounds on the Threshold



References

For more info, see e.g.

- R. Koetter and P. O. Vontobel, "On the block error probability of LP decoding of LDPC codes", Proc. Inaugural Workshop of the Center for Information Theory and its Applications, UCSD, La Jolla, CA, USA, Feb. 6-10, 2006. [http://www.arxiv.org/abs/cs.IT/0602086]

- P. O. Vontobel and R. Koetter, "Bounds on the threshold of linear programming decoding", Proc. IEEE Inform. Theory Workshop, Punta Del Este, Uruguay, Mar. 13-16, 2006. [http://www.arxiv.org/abs/cs.IT/0602087]