On Vector Witsenhausen’s Counterexample
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Motivation: Control Systems
- LQG systems:
  - Plant dynamics linear
  - Quadratic cost constraints on controller actions
  - Initial conditions and observation noises are Gaussian
- Objective: optimal control actions that collectively stabilize the plant

Centralized vs. Decentralized Control
- Centralized Controller
  - Observes plant state directly
  - Linear controller optimum for LQG system
- Decentralized Controller
  - Observes plant state indirectly
  - Multiple controllers
  - Decentralized controller optimum?

Witsenhausen’s Counterexample
- Figure: Example of Witsenhausen’s counterexample (image courtesy [Grover-Sahai 10])
- [Witsenhausen 68] provides toy counterexample
  - Nonlinear strategies can outperform linear [Witsenhausen 68]
  - By an unbounded factor [Mitter-Sahai 99]
  - Finding optimal strategy is NP-hard [Papadimitriou-Tsitsiklis 84]
  - Semi-exhaustive search techniques [Baglietto-Parsini-Zoppoli 01], [Lee-Lau-Ho 01], [Lee-Marden-Shamma 09]

Scalar Witsenhausen Problem
- Scalar modified state estimation problem:
  - $S \sim N(0, \sigma^2)$
  - $X \sim N(0, 1)$
  - $Z \sim N(0, 1)$
  - Minimize $k^2 E(Z^2) + E(X^2)

Proposed Lower Bound
- $D^* \text{ is lower bounded by}
  \begin{align*}
  D^* \geq \min_{p(v|s), p(u|s, v)} E(d(X, \hat{X}))
  \end{align*}
where minimum is over all pmfs $p(v|s), p(u|s, v)$ and function $\hat{x}(v, y)$ such that

\[ l(U, S, Y) \geq l(V, Y, S). \]

Computing Lower Bound
\begin{align*}
  l(U, S, Y) - l(V, Y, S) &= h(Y) + h(S - \alpha \hat{X} - \beta Y|V, Y) - h(S) - h(Z) \\
  \geq h(Y) + h(S - \alpha \hat{X} - \beta Y) - h(S) - h(Z) \\
  &\geq h(Y) + h(S - \alpha \hat{X} - \beta Y) - h(S) - h(Z)
\end{align*}

Optimality of Lower Bound
- Lower bound will be tight if $\alpha$ and $\beta$ is selected such that
  - $S - \alpha \hat{X} - \beta Y$ is orthogonal to the plane containing $(V, Y)$ (from (a)), and
  - $S - \alpha \hat{X} - \beta Y$ is a scalar multiple of $\hat{X} - X$ (from (b)).

Achievable Strategy
- Achievable strategy is coding scheme in [Grover-Sahai 10]
- A combination of linear coding and DPC is optimal
- $U = U_{dpc} + U_{inc}$, where $U_{inc} = -BS$ and $U_{dpc}$ is independent of $S$ and $Z$
- $V = U_{dpc} + (1 - b)X_0$, $a$ is amplification factor
- Choose $\hat{X}(V, Y) = E(X|V, Y)$

Concluding Remarks
- Studied asymptotic Witsenhausen’s counterexample
- Determined minimum distortion by improving prior lower bounds
- Showed optimality of prior achievable strategy (linear coding+DPC)
- Improved constant gap result for scalar Witsenhausen!